ANALYSIS OF MARTIN-HARRINGTON THEOREM IN HIGHER ORDER ARITHMETIC

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DECLARATION

I hereby declare that this thesis is my original work and it has been written by me in its entirety.

I have duly acknowledged all the sources of information which have been used in the thesis.

This thesis has also not been submitted for any degree in any university previously.

Cheng Yong

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Contents

Summary

The main effort in this thesis is to answer some questions from Professor W.Hugh Woodin about Martin-Harrington theorem. The boldface Martin-Harrington theorem says that $Det(\underset{\sim}{\Sigma}^1_1)$ if and only if for any real $x, x^\sharp$ exists and the lightface Martin-Harrington theorem says that $Det(\Sigma^1_1)$ if and only if $0^\sharp$ exists.

Harrington's theorem "$Det(\Sigma^1_1)$ implies $0^\sharp$ exists" is proved in two steps: first show that "$Det(\Sigma^1_1)$ implies **Harrington's** $\star$" and then derive the existence of $0^\sharp$ from **Harrington's** $\star$ by the use of Silver's theorem. We observe that "$Z_2 + Det(\Sigma^1_1)$ implies **Harrington's** $\star$". The first question from Professor W.Hugh Woodin is "whether Z_2+**Harrington's** $\star$ implies $0^\sharp$ exists". We show that Z_2+ **Harrington's** $\star$ does not imply $0^\sharp$ exists. The second question from Professor W.Hugh Woodin is "whether Z_3+**Harrington's** $\star$ implies $0^\sharp$ exists". We show that Z_3+ **Harrington's** $\star$ does not imply $0^\sharp$ exists. As a corollary of "Z_4+ **Harrington's** $\star$ implies $0^\sharp$ exists", Z_4 is the minimal system in higher order arithmetic to prove "**Harrington's** $\star$ implies $0^\sharp$ exists".

Finally, this thesis examines the question "whether Martin-Harrington theorem is provable in Z_2" from Professor W.Hugh Woodin. We observe that the direction from $0^\sharp$ to determinacy in Martin-Harrington theorem is

provable in Z_2. So the question reduces to "whether boldface and lightface Harrington's theorem are provable in Z_2". As a corollary of "Z_4+ **Harrington's** $\star$ implies $0^\sharp$ exists", lightface Harrington's theorem is provable in Z_4. We show that boldface Harrington's theorem is provable in Z_2.

Key Words: Martin-Harrington theorem, Harrington's theorem, **Harrington's** $\star$, $0^\sharp$, almost disjoint forcing, Baumgartner's forcing, weakly reflecting property, strong reflecting property, Z_2, Z_3, Z_4.

Chapter 1

Introduction

1.1 Notations and definitions

Unless otherwise specified, we use $\alpha, \beta, \gamma, \delta \cdots$ to denote ordinals and κ, λ, $\mu, \nu \cdots$ to denote infinite cardinals. As usual, $\omega = \{0, 1, \cdots\}$ and $\mathbb{R} = \omega^\omega$. Elements of $\mathbb{R}$ or ω^ω or 2^ω are called reals. In this thesis, countable set is always assumed to be infinite. $cf(\gamma)$ denotes the cofinality of γ and γ^+ denotes the least cardinal greater than γ. Ord denotes the class of ordinals, V the universe of sets, V_α the set of sets of rank less than α and $trc(x)$ the transitive closure of x (the smallest transitive set $\supseteq x$). $A \setminus B$ denotes set subtraction. For $X \subseteq Ord, o.t.(X)$ denotes its order type. For a set $x, |x|$ denotes its cardinality and $\mathcal{P}(x)$ its power set. For a function $f, dom(f)$ denotes its domain, $ran(f)$ its range, $f``X = \{f(y) \mid y \in X\}$, $f \restriction X = f \cap (X \times V)$ (the restriction of f to X) and $f^{-1}(X) = \{y \in dom(f) \mid f(y) \in X\}$. If M is a transitive set, $Ord(M)$ denotes $Ord \cap M$. For uncountable cardinal κ,

$H_\kappa = \{x \mid |trcl(x)| < \kappa\}$. HC denotes H_{ω_1}. A cardinal κ is strong limit iff for any $\lambda < \kappa, 2^\lambda < \kappa$. For a set $X, [X]^\kappa = \{Y \subseteq X \mid |Y| = \kappa\}$ and $[X]^{<\kappa} = \{Y \subseteq X \mid |Y| < \kappa\}$ (we often write $X^{<\omega}$ for $[X]^{<\omega}$). ω^ω and $\mathbb{R}$ both denote the set of all reals. κ is weakly inaccessible if κ is a uncountable regular limit cardinal. κ is inaccessible if κ is a uncountable regular cardinal and for any $\lambda < \kappa, 2^\lambda < \kappa$.

$\mathfrak{L}_{st}$ denotes the language of set theory: first-order predicate calculus with equality and the binary predicate symbol $\in$. In this language AC denotes the Axiom of Choice, CH the Continuum Hypothesis, and GCH the Generalized Continuum Hypothesis. ZFC denotes Zermelo-Fraenkel Set Theory with the Axiom of Choice in $\mathfrak{L}_{st}$. ZF denotes Zermelo-Fraenkel Set Theory without the Axiom of Choice. ZF^- denotes ZF with the Power Set Axiom deleted. Similarly for ZFC^-. For a formula φ, $\ulcorner\varphi\urcorner$ denotes its code according to some contextually established arithmetization.

Definition 1.1.1 *(i)* $Z_2 = ZFC^- +$ *Any set is Countable.*

(ii) $Z_3 = ZFC^- + \mathcal{P}(\omega)$ *exists + Any set is of cardinality* $\leq |\mathbb{R}|$.

(iii) $Z_4 = ZFC^- + \mathcal{P}(\mathcal{P}(\omega))$ *exists + Any set is of cardinality* $\leq |\mathcal{P}(\mathbb{R})|$.

Z_2, Z_3 and Z_4 are the corresponding axiomatic systems in the language of set theory for Second Order Arithmetic (SOA), Third Order Arithmetic and

Fourth Order Arithmetic. Under $CH, Z_3 = ZFC^- + \mathcal{P}(\omega)$ exists + Any set is of cardinality $\leq \omega_1$. Assuming $GCH, Z_4 = ZFC^- + \mathcal{P}(\mathcal{P}(\omega))$ exists + Any set is of cardinality $\leq \omega_2$. Similarly, we can define Z_n for $n > 4$.

KP (Kripke-Platek Set Theory) consists of axioms of BS(Basic Set Theory)[1] together with Δ_0-collection schema:

$$\forall \vec{a}(\forall x \exists y \varphi(x, y, \vec{a}) \to \forall u \exists v (\forall x \in u)(\exists y \in v)\varphi(x, y, \vec{a}))$$

for some Δ_0-formula $\varphi(x, y, \vec{a})$.

A transitive set M is said to be admissible if and only if $M \models KP$. α is an admissible ordinal if $L_\alpha \models KP$. For any set X, α is X-admissible if $L_\alpha[X] \models KP$; ω_1^X is the least X-admissible ordinal and $L_{\omega_1^X}[X]$ is the least admissible set containing ω and X as elements. The main reference about admissibility is [1].

Convention For admissible ordinal α, we always assume that $\alpha > \omega$.

$M \preceq_n N$ means M is a Σ_n elementary submodel of N. i.e. for any Σ_n formula φ with parameters from M, $M \models \varphi$ if and only if $N \models \varphi$. A function is $\Sigma_n(L_\alpha)$ if and only if it is Σ_n definable over L_α with parameters from L_α.

[1] BS consists of the following schema of axioms: Extensionality, Pairing, Union, Infinity, Cartesian Product, Induction Schema and Σ_0 comprehension schema. For details, see [2].

Fact 1.1.2 *([2])* *α is admissible if and only if there is no $\Sigma_1(L_\alpha)$ map f which maps some $\beta < \alpha$ cofinally into α. α is X-admissible if and only if there is no $\Sigma_1(L_\alpha[X])$ map f which maps some $\beta < \alpha$ cofinally into α.*

Fact 1.1.3 *([13])* *If A is an admissible set and $R \in A$ is a well ordering, then there exists $\alpha \in A \cap Ord$, and a function $f \in A$ such that f maps R isomorphically onto $\in \restriction \alpha$.*

For $n_0, \cdots, n_{k-1} \in \omega$, we use $\langle n_0, \cdots, n_{k-1} \rangle$ to denote the natural number encoding $(n_0, \cdots, n_{k-1})$ via a recursive bijection (which we fix throughout) between ω^k and ω. We regarded reals as codes for relations. Any $x \in \omega^\omega$ encodes a binary relation E_x on ω given by $(m, n) \in E_x$ iff $x(\langle m, n \rangle) = 0$.

$WF = \{x \in \omega^\omega \mid E_x \text{ is well founded}\}$. $WO = \{x \in \omega^\omega \mid E_x \text{ is well ordered}\}$.

For $x \in WF$, $|x|$ denotes the rank of the well founded relation E_x. For

$$\alpha < \omega_1, WO_{<\alpha} = \{x \in WO \mid |x| < \alpha\}.$$

Define

$$\omega_1^{CK} = \sup\{|x| \mid x \in WF \text{ and the graph of } x \text{ is recursive}\}.$$

ω_1^{CK} is the least non-recursive ordinal. Similarly, for real x, we can define ω_1^x.

Fact 1.1.4 *([13], [1])* *Given real x, ω_1^x is the least x-admissible ordinal, the least admissible ordinal which is not recursive in x, the least ordinal which* *is not the order type of a well ordering on ω which is recursive in x and the least ordinal which is not the order type of a $\Delta^1_1(x)$ well ordering on ω.*

Fact 1.1.5 *([15]) (Boundedness theorem for $\underset{\sim}{\Sigma}^1_1$ (Σ^1_1) set)*
If $A \subseteq WO$ is $\underset{\sim}{\Sigma}^1_1$, then there is an $\alpha < \omega_1$ such that $A \subseteq WO_{<\alpha}$; if $A \subseteq WO$ is Σ^1_1, then there is an $\alpha < \omega_1^{CK}$ such that $A \subseteq WO_{<\alpha}$.

Suppose $\mathcal{M} = (M, E)$ is a model in the language $\mathfrak{L}_{st}$.

(1) $Ord^{\mathcal{M}}$ denotes the class of all ordinals of (M, E) and $o(\mathcal{M})$ denotes the least ordinal not in M.

(2) Define $wfp(\mathcal{M}) = \{x \in M \mid$ the restriction of E to $\{y \in M \mid yEx\}$ is well founded$\}$. $wfp(\mathcal{M})$ is called the well founded part of (M, E). We usually assume that $wfp(\mathcal{M})$ is transitive.

(3) $\mathcal{M}$ is an ω-model if $\omega \in wfp(\mathcal{M})$ and the restriction of E to $wfp(\mathcal{M})$ is the membership relation.

(4) The ordinal standard part of $\mathcal{M}$ (denoted by $osp(\mathcal{M})$) is the least ordinal not in $wfp(\mathcal{M})$. Equivalently, $osp(\mathcal{M})$ is the greatest ordinal α such that $(Ord^{\mathcal{M}}, E \restriction Ord^{\mathcal{M}})$ has an initial segment of order type α.

Fact 1.1.6 *([13]) Suppose $\mathcal{M} = (M, E)$ is an ω-model of KP. Then $wfp(\mathcal{M}) \models KP$, $osp(\mathcal{M})$ is an admissible ordinal and $osp(\mathcal{M})$ is not definable in $\mathcal{M}$. Similarly for ω-model of ZFC.*

Definition 1.1.7 *For a set of reals $A \subseteq \omega^\omega$, G_A is the game as follows in which player I and player II alternately play natural numbers.*

$$G_A : \begin{array}{c|cccccccccc} I & n_0 & & n_2 & \cdots & n_{2t} & & & \cdots & \\ \hline II & & n_1 & & n_3 & \cdots & n_{2t+1} & & & \cdots \end{array}$$

Let $x = (n_0, n_1, n_2, \cdots, n_{2t}, n_{2t+1} \cdots) \in \omega^\omega$. x is called a play of the game. We say that Player I wins G_A if $x \in A$; otherwise Player II wins G_A.

(i) A strategy for Player I is a function $\sigma : \bigcup_{i\in\omega} \omega^{2i} \to \omega$. Let $\sigma * y$ be the real produced when Player I follows σ and Player II plays y. σ is a winning strategy for Player I in G_A iff for all $y \in \omega^\omega, \sigma * y \in A$. i.e. Player I always wins G_A by following σ no matter how Player II plays. The corresponding notions for Player II are defined similarly.

(ii) $\sigma * y$ is the play in which Player II plays y against σ and $x * \tau$ is the play in which Player I plays x against τ. $x * y$ is the resulting real in a play in which Player I plays x and Player II plays y. In this case we let $(x * y)_I = x$ and $(x * y)_{II} = y$.[2] If σ is a strategy for Player I and τ is a strategy for Player II we write $\sigma * \tau$ for the real produced by playing the strategies against one another.

(iii) Given a real x, if σ is a winning strategy in G_A for player I, we say that

[2] $(\sigma * y)_I$ is the real Player I plays in a play in which Player I follows the strategy σ against II's play of y. Similarly for $(x * \tau)_{II}$.

x is consistent with σ if $x = \sigma * y$ for some $y \subseteq \omega$. Similarly, if τ is a winning strategy in G_A for player II, we say that x is consistent with τ if $x = y * \tau$ for some $y \subseteq \omega$.

(iv) G is determined (denoted by $Det(G)$) iff one of the players has a winning strategy in game G. A is determined (denoted by $Det(A)$) iff G_A is determined.

A partial order (p.o.) is a partially ordered set $\langle P, \leq \rangle$ that has a maximum element denoted by $\mathbf{1}$, and for $p, q \in P$,

$$p \leq q \leftrightarrow p \text{ extends or refines } q\,.$$

For a p.o. $P \in M, G$ is P-generic over M if and only if $G \subseteq P, G$ is a filter and $\forall D \subseteq P((D \text{ is dense in } P \wedge D \in M) \rightarrow G \cap D \neq \emptyset)$. Our notations about forcing are standard (see [9]).

Definition 1.1.8 *(i) A partial order P is κ-closed if and only if whenever $\lambda < \kappa$ and $\{p_\alpha : \alpha < \lambda\} \subseteq P$ with $p_\beta \leq p_\alpha$ for $\alpha < \beta < \lambda$, there exists $p \in P$ such that for any $\alpha < \lambda, p \leq p_\alpha$.*

(ii) A partial order P satisfies κ chain condition(κ-c.c) if and only if for any antichain A in $P, |A| < \kappa$. P has countable chain condition(c.c.c) if it is ω_1-c.c.

(iii) A partial order P is κ-distributive if and only if whenever $\gamma < \kappa$ and D_α is dense open for each $\alpha < \gamma$, $\bigcap_{\alpha<\gamma} D_\alpha$ is dense.

Fact 1.1.9 *([9], [12],[8])*

(1) If $(P, <)$ is κ-closed, then it is κ-distributive.

(2) If λ is a cardinal and $(P, <)$ is λ-closed, then P preserves cofinality $\leq \lambda$ and hence preserves cardinals $\leq \lambda$.

(3) If λ is a cardinal and $(P, <)$ is λ-c.c, then P preserves cofinality $\geq \lambda$. If moreover λ is a regular cardinal, then P preserves cardinals $\geq \lambda$.

(4) $(P, <)$ is κ-distributive if and only if every function $f : \alpha \to V$ in the generic extension with $\alpha < \kappa$ is in the ground model.

(5) If $(P, <)$ is κ-distributive, then all cardinals $\leq \kappa$ in V remains cardinals in $V[G]$.

Definition 1.1.10 *(Shelah) Suppose P is a forcing notion, $\kappa > 2^{|P|}$ is an uncountable cardinal and $M \prec H_\kappa$ such that $|M| = \omega$ and $P \in M$. We say that a condition $p \in P$ is (M, P)-generic if and only if for every dense (antichain, predense)$D \subseteq P$ with $D \in M, D \cap M$ is predense below p (i.e. for all $q \leq p$, there exists $d \in D \cap M$ such that q is compatible with d).*

Definition 1.1.11 *(Shelah)* *A poset P is proper if and only if for every regular uncountable cardinal $\kappa > 2^{|P|}$, for any $M \prec H_\kappa$ such that $|M| = \omega$ and $P \in M$, every $p \in P \cap M$ has an extension $q \leq p$ such that q is an (M, P)-generic condition.*

Fact 1.1.12 *(Baumgartner, Jech, Shelah, [9], [8])* *Given a poset P, the following are equivalent:*

(1) P is proper.

(2) For some regular uncountable cardinal $\kappa > 2^{|P|}$, for any $M \prec H_\kappa$ such that $|M| = \omega$ and $P \in M$, every $p \in P \cap M$ has an extension $q \leq p$ such that q is an (M, P)-generic condition.

(3) For every uncountable cardinal κ, P preserves stationary subsets of $[\kappa]^\omega$.

(4) For some (any) regular $\kappa > 2^{|P|}$, $\{M \prec H_\kappa \mid |M| = \omega, P \in M$ and $\forall p \in P \cap M \, \exists q \leq p(q$ is (M, P)-generic$)\}$ contains a club subset of $[H_\kappa]^\omega$.

Definition 1.1.13 *Let A be an uncountable set and $C \subseteq [A]^\omega$.*

(i) C is unbounded if for any $x \in [A]^\omega$, there is $y \in C$ such that $x \subseteq y$.

(ii) C is closed if for every chain $x_0 \subseteq x_1 \subseteq \cdots \subseteq x_n \subseteq \cdots$ in $C, \bigcup_{n\in\omega} x_n \in C$.

(iii) C *is a club on* $[A]^\omega$ *if* C *is closed and unbounded.*

(iv) $S \subseteq [A]^\omega$ *is stationary if for any club* C *on* $[A]^\omega, S \cap C \neq \emptyset$.

(v) For $F : A^{<\omega} \to A, x \subseteq A$ *is closed under* F *if and only if* $F``(x^{<\omega}) \subseteq x$.

Define $C_F = \{x \in [A]^\omega \mid x \text{ is closed under } F\}$.

Note that if $|A| = \omega_1$, then the concept of club and stationary coincides essentially with the usual concept of club and stationary. We can characterize club and stationary sets in terms of functions $F : A^{<\omega} \to A$.

Fact 1.1.14 *([8])*

(1) If $F : A^{<\omega} \to A$*, then* C_F *is a club.*

(2) For every club C *on* $[A]^\omega$*, there exists a function* $F : A^{<\omega} \to A$ *such that* $C_F \subseteq C$.

(3) $S \subseteq [A]^\omega$ *is stationary if and only if for every function* $F : A^{<\omega} \to A$*, there is* $x \in S$ *such that* x *is closed under* F.

(4) If $A \subseteq B$ *and* C *is a club in* $[B]^\omega$*, then* $C \restriction A = \{x \cap A | x \in C\}$ *contains a club in* $[A]^\omega$.

The theory of $0^\sharp$ in ZFC was developed in [2]. In fact the theory of $0^\sharp$ can be developed in Z_2 and we can define $0^\sharp$ in Z_2.

Definition 1.1.15 *For $\mathcal{M}$ a structure and X a subset of the domain of $\mathcal{M}$ linearly ordered by $<$ (not necessarily a relation of $\mathcal{M}$), $\langle X, < \rangle$ is a set of indiscernibles for $\mathcal{M}$ if and only if for every formula $\varphi(v_1, \cdots, v_n)$ in the language of $\mathcal{M}$ with $x_1 < \cdots < x_n$ and $y_1 < \cdots < y_n$ all in X, we have*

$$\mathcal{M} \models \varphi[x_1, \cdots, x_n] \leftrightarrow \mathcal{M} \models \varphi[y_1, \cdots, y_n].$$

i.e. For each $n \in \omega$ all increasing n-tuples from X have the same first order properties in $\mathcal{M}$.

Definition 1.1.16 *Let $\mathfrak{L}^*_{st}$ be $\mathfrak{L}_{st}$ augmented by constants $\{c_k \mid k \in \omega\}$. The theory of the structure $\langle L_\alpha, \in, \gamma_k \rangle_{k \in \omega}$ in $\mathfrak{L}^*_{st}$ is called an EM (Ehrenfeucht-Mostowski) set, where α is a countable limit ordinal $> \omega$ and $\{\gamma_k \mid k \in \omega\}$ is a set of ordinal indiscernibles for $\langle L_\alpha, \in \rangle$ indexed in increasing order.*

Definition 1.1.17 *Suppose Σ is an EM set and α is an infinite countable ordinal. $(\mathcal{A}, H)$ is called a (Σ, α) model if and only if*

(a) $\mathcal{A} = \langle A, E \rangle$ is a model of $ZF + V = L$;

(b) $H \subseteq Ord^{\mathcal{A}}$ is a set of ordinal indiscernible for $\mathcal{A}$ with order type α;

(c) $\mathcal{A} = \mathcal{A} \upharpoonright H$;

(d) Σ is a set of $\mathfrak{L}_{st}$-formulas which are valid in $\mathcal{A}$ on increasing sequences from H.

Definition 1.1.18 *(a) An EM set Σ is cofinal if and only if it contains all formulas in the form*

$$\text{“}t(v_0, \cdots, v_{n-1}) \in Ord \to t(v_0, \cdots, v_{n-1}) < v_n\text{”}$$

for any Skolem term t.

(b) An EM set Σ is remarkable if and only if for any Skolem term t, if the formula

$$\text{“}t(v_0, \cdots, v_{n-1}, v_n, \cdots, v_{n+m}) < v_n\text{”}$$

is in Σ, then the formula

$$\text{“}t(v_0, \cdots, v_{n-1}, v_n, \cdots, v_{n+m}) = t(v_0, \cdots, v_{n-1}, v_{n+m+1}, \cdots, v_{n+2m+1})\text{”}$$

is in Σ.

(c) An EM set Σ is well founded if and only if for any infinite countable ordinal α, the (Σ, α) model is well founded.

Fact 1.1.19 *([2]) Let Σ be an EM set, for any infinite countable ordinal α, there is an unique (up to isomorphism) (Σ, α) model.*

We will be interested in well founded (Σ, α) model. If Σ is a well founded EM set, then for any infinite countable ordinal α, there is an unique transitive (Σ, α) model and we denote it by $\mathcal{M}(\Sigma, \alpha)$.

Proposition 1.1.20 *If there exists a well founded remarkable cofinal EM set, then it is unique.*

Proof Let Σ be a well founded remarkable cofinal EM set. Let (L_α, H) be the unique transitive (up to isomorphism) (Σ, ω_1^{CK}) model. Let $(h_\theta : \theta < \omega_1^{CK})$ be an increasing enumeration of H, then

$$\varphi(v_0, \cdots, v_n) \in \Sigma \leftrightarrow L_\alpha \models \varphi[h_0, \cdots, h_n].$$

So Σ is unique. $\square$

Definition 1.1.21 *The unique well founded remarkable cofinal EM set, if it exists, is denoted by* $0^\sharp$.

Fact 1.1.22 *([2])* $0^\sharp$ *is a* Π^1_2 *singleton. i.e.* $0^\sharp$ *is an unique solution of a* Π^1_2 *predicate. As a corollary,* $0^\sharp$ *is a* Δ^1_3 *real and "*$0^\sharp$ *exists" is* Σ^1_3.

Theorem 1.1.23 *([2], [3])* (Z_3) $0^\sharp$ *exists if and only if* L_{ω_1} *has an uncountable set of indiscernibles.*

Definition 1.1.24 *Suppose that* I *is a set of* $<$*-indiscernibles over a structure* $\mathcal{M}$. *Then the indiscernibility type* Σ *of* I *is defined as the set of all formulas* $\varphi(v_1, \cdots, v_n)$ *such that* $\mathcal{M} \models \varphi[i_1, \cdots, i_n]$ *where* $i_1, \cdots, i_n \in I$ *and* $i_1 < \cdots < i_n$.

In fact, in the proof of Theorem 1.1.23, the existence of uncountably many indiscernibles is really not required to show the existence of $0^\sharp$. It suffices to know only that sets of indiscernibles of every order type $\alpha < \omega_1$ can be found, all of which have the same indiscernibility type over the given structure.

Corollary 1.1.25 *Suppose that there exists a set of formulas Σ in $\mathfrak{L}_{st}$ such that for every $\alpha < \omega_1$ there exists a set I_α of indiscernibles for L_{ω_1} such that o.t.$(I_\alpha) = \alpha$ and*

$$\Sigma = \{\varphi(v_1, \cdots, v_n) \mid L_{\omega_1} \models \varphi[i_1, \cdots, i_n]\}$$

where $i_1, \cdots, i_n \in I_\alpha$ and $i_1 < \cdots < i_n$. Then $0^\sharp$ exists.

Proposition 1.1.26 *([2], [3], [11]) (Z_3) The following are equivalent.*

(1) $0^\sharp$ exists.

(2) L_{ω_1} has an uncountable set of indiscernibles.

(3) There exists an uncountable subset $C \subseteq \omega_1$ such that for any formula φ and for any two increasing sequences $\xi_0 < \cdots < \xi_{n-1}$ and $\xi'_0 < \cdots < \xi'_{n-1}$ of elements from C, we have

$$L_{\omega_1} \models \varphi[\xi_0, \cdots, \xi_{n-1}] \leftrightarrow L_{\omega_1} \models \varphi[\xi'_0, \cdots, \xi'_{n-1}].$$

(4) For each formula φ, there exists a closed unbounded subset $C = C_\varphi$ of ω_1 such that either

(a) $L_{\omega_1} \models \varphi[\xi_0, \cdots, \xi_{n-1}]$ *for any increasing sequence* $\xi_0 < \cdots < \xi_{n-1}$ *of elements from* C, *or*

(b) $L_{\omega_1} \models \neg\varphi[\xi_0, \cdots, \xi_{n-1}]$ *for any increasing sequence* $\xi_0 < \cdots < \xi_{n-1}$ *of elements from* C.

(5) There exists a set of formulas Σ *in* $\mathfrak{L}_{st}$ *such that for every* $\alpha < \omega_1$ *there exists a set* I_α *of indiscernibles for* L_{ω_1} *such that o.t.*$(I_\alpha) = \alpha$ *and*

$$\Sigma = \{\varphi(v_1, \cdots, v_n) \mid L_{\omega_1} \models \varphi[i_1, \cdots, i_n]\}$$

where $i_1, \cdots, i_n \in I_\alpha$ *and* $i_1 < \cdots < i_n$.

(6) There exists a well founded, cofinal and remarkable EM set.

We give some remarks about the relationship between Z_2 and SOA (Second Order Arithmetic).

Definition 1.1.27 *(i) Let* M, N *be two structures respectively in the language of* $\mathcal{L}_1$ *and* $\mathcal{L}_2$. M *and* N *are bi-interpretable if and only if there exists a recursive function* $\varphi \mapsto \varphi^*$ *such that*

$$M \models \varphi \Leftrightarrow N \models \varphi^*,$$

where φ *is a formula in* $\mathcal{L}_1$ *and* φ^* *in* $\mathcal{L}_2$.

(ii) Suppose T_1 *and* T_2 *are recursively enumerable axiom systems. We say that* T_1 *is interpretable in* T_2 $(T_1 \leq T_2)$ *if there is a translation* τ *from*

the language of T_1 to the language of T_2 such that, for each sentence φ of the language of T_1, if $T_1 \vdash \varphi$ then $T_2 \vdash \tau(\varphi)$. Let $T_1 < T_2 \leftrightarrow T_1 \leq T_2 \wedge T_2 \not\leq T_1$ and $T_1 \equiv T_2 \leftrightarrow T_1 \leq T_2 \wedge T_2 \leq T_1$. If $T_1 \equiv T_2$, we say that T_1 and T_2 are bi-interpretable.

Since "any set is countable" is equivalent to $V = HC$, $Z_2 = ZFC^- + V = HC$. SOA is defined in the language of analysis and Z_2 is defined in the language of set theory. Note that $(\omega, \mathcal{P}(\omega), +, \cdot, \in) \models SOA$, $V_{\omega+1} \models Z_2$ and $HC \models Z_2$.

Fact 1.1.28 *(Folklore)*

(1) SOA and Z_2 are bi-interpretable.

(2) Structures $(\omega, \mathcal{P}(\omega), +, \cdot, \in)$, $(V_{\omega+1}, \in)$ and $(HC, \in)$ are bi-interpretable.

Definition 1.1.29 ***Harrington's*** $\star$ *denotes the following statement:*

$$\exists x \in 2^\omega \forall \alpha < \omega_1 (\alpha \text{ is } x\text{-admissible} \rightarrow \alpha \text{ is an } L\text{-cardinal}).$$

Fact 1.1.30 *For $\alpha < \omega_1, L \models$ "α is a cardinal" if and only if $\forall \beta < \omega_1 (\alpha \in L_\beta \rightarrow L_\beta \models$ "α is a cardinal").*

Proof Suppose $L \models \alpha$ is not a cardinal. Then $L \models \exists \beta < \alpha \exists f (f$ is a surjection from β to $\alpha)$. Let β, f be the witness in L. Since $\alpha < \omega_1$, $f \in L_{\omega_1}$. Take $\gamma < \omega_1$ such that $\alpha < \gamma$ and $f \in L_\gamma$. Then $L_\gamma \models \alpha$ is not a cardinal. □

So for $\alpha < \omega_1$, "α is an L-cardinal" is Π^1_2. Hence "**Harrington's** $\star$" is Σ^1_3. Note that $Det(\Sigma^1_1)$ and "$0^\sharp$ exists" are also Σ^1_3 statements.

1.2 Thesis Problems

For the last three decades, much work has been done on the relationship between large cardinal and determinacy hypothesis, especially the large cardinal-determinacy correspondence. The first result in this line was proved by Martin and Harrington.

Theorem 1.2.1 *(Martin-Harrington theorem, [9])*

(i) (Boldface version) (ZF) $Det(\underset{\sim}{\Sigma}^1_1)$ *if and only if for any real* $x, x^\sharp$ *exists.*

(ii) (Lightface version) (ZF) $Det(\Sigma^1_1)$ *if and only if* $0^\sharp$ *exists.*

Martin-Harrington theorem 1.2.1 is a milestone for the latter investigation of correspondence between large cardinal and determinacy hypothesis. This remarkable equivalence is an unexpected confluence that bolstered both the large cardinal and the determinacy theory and motivated further research on the relationship between large cardinal and determinacy hypothesis.

Theorem 1.2.2 *(Silver, [5])* *(ZF)* *Suppose* $x \subseteq \omega$ *and for any* $\alpha < \omega_1$, *if* α *is* x*-admissible, then* α *is an* L*-cardinal. Then* $0^\sharp \in L[x]$. *So* ***Harrington's*** $\star$ *implies* $0^\sharp$ *exists.*

Theorem 1.2.3 *(Silver, Solovay,[9])* *Assume $0^\sharp$ exists. Let I be the class of Silver indiscernibles. If α is $0^\sharp$-admissible, then I is unbounded in α. As a corollary, for any ordinal α, if α is $0^\sharp$-admissible, then α is an L-cardinal. So, $0^\sharp$ exists implies **Harrington's** $\star$.*

So in ZF we have

$$Det(\Sigma_1^1) \leftrightarrow \textbf{Harrington's } \star \leftrightarrow 0^\sharp \text{ exists.}$$

Harrington's proof of "$Det(\Sigma_1^1)$ implies $0^\sharp$ exists" in ZF is done in two steps.

First Step $Det(\Sigma_1^1)$ implies **Harrington's** $\star$.

Second Step **Harrington's** $\star$ implies $0^\sharp$ exists by Silver's theorem 1.2.2.

In fact, all known proofs of "$Det(\Sigma_1^1)$ implies $0^\sharp$ exist" in ZFC use Silver's theorem 1.2.2. We observe that the first step "$Det(\Sigma_1^1)$ implies **Harrington's** $\star$" is provable in Z_2. For different proofs of "Z_2 + $Det(\Sigma_1^1)$ implies **Harrington's** $\star$", see [5], [13], [16] and W.Hugh Woodin's proof in Section 3.2. The next natural question is:

Question 1.2.4 *(W.Hugh Woodin)* *Whether Z_2+**Harrington's** $\star$ implies $0^\sharp$ exists?*

If the answer is positive, then "$Det(\Sigma_1^1)$ implies $0^\sharp$ exists" is provable in Z_2. If the answer is negative and "$Det(\Sigma_1^1)$ implies $0^\sharp$ exists" is provable in Z_2, then there must be a new and different proof of "$Det(\Sigma^1_1)$ implies $0^\sharp$ exists" without the use of **Harrington's** $\star$ to derive the existence of $0^\sharp$. In this thesis, we prove that Z_2+**Harrington's** $\star$ does not imply $0^\sharp$ exists.

Question 1.2.5 *(W.Hugh Woodin) Whether Z_3+**Harrington's** $\star$ implies $0^\sharp$ exists?*

If the answer is positive, then Z_3 is the minimal system in higher order arithmetic to prove "**Harrington's** $\star$ implies $0^\sharp$ exists" and "$Det(\Sigma^1_1)$ implies $0^\sharp$ exists" is provable in Z_3. If the answer is negative, then by Theorem 2.0.3 Z_4 is the minimal system in higher order arithmetic to prove "**Harrington's** $\star$ implies $0^\sharp$ exists".[3]

Convention Throughout this thesis, lightface Harrington's theorem refers to the theorem "$Det(\Sigma^1_1)$ implies $0^\sharp$ exists" and boldface Harrington's theorem refers to the theorem "$Det(\underset{\sim}{\Sigma}^1_1)$ implies for any real $x, x^\sharp$ exists". Harrington's theorem refers to these two versions.

Question 1.2.6 *(W.Hugh Woodin) Whether Martin-Harrington theorem is provable in Z_2?*

[3] In this thesis we focus on the provability strength of the statement "**Harrington's** $\star$ implies $0^\sharp$ exists" in higher order arithmetic. However, generally we did not intend to advocate a research program to examine the provability strength of every known theorem in set theory in higher order arithmetic. In this thesis, we examine higher order arithmetic in the base theory ZFC or $ZFC+$large cardinals.

We observe that the direction from $0^\sharp$ to determinacy in Martin-Harrington theorem is provable in Z_2. So the question reduces to whether Harrington's theorem is provable in Z_2.

1.3 The structure of the thesis

This thesis consists of four Chapters:

- In Chapter 1, we introduce thesis problems, outline the structure of the thesis and review notations and definitions used throughout the thesis.
- In Chapter 2, we answer Question 1.2.4 and Question 1.2.5. As a corollary, Z_4 is the minimal system to prove "**Harrington's** $\star$ implies $0^\sharp$ exists" in higher order arithmetic.
- In Chapter 3, we prove boldface Martin-Harrington theorem in Z_2 and present W.Hugh Woodin's proof of Harrington's theorem.
- In Chapter 4, we give a summary of main results in this thesis and propose problems for future research.

Chapter 2

Minimal system for “Harrington’s $\star$ implies $0^\sharp$ exists” in higher order arithmetic

In this chapter we prove that Z_2+**Harrington’s** $\star$ does not imply $0^\sharp$ exists and Z_3+**Harrington’s** $\star$ does not imply $0^\sharp$ exists. These answer Question 1.2.4 and Question 1.2.5. As a corollary, Z_4 is the minimal system to show that “**Harrington’s** $\star$ implies $0^\sharp$ exists”.

Silver first proved in ZF that **Harrington’s** $\star$ implies $0^\sharp$ exists.

Theorem 2.0.1 *(Silver)* (ZF) *Suppose a is a real such that for any $\alpha < \omega_1$, if α is a-admissible, then α is an L-cardinal. Then $0^\sharp \in L[a]$. So **Harrington’s** $\star$ implies $0^\sharp$ exists.*

Proof We work in $L[a]$. Take a submodel M of $L_{\omega_3}[a]$ such that

(1) $M \prec L_{\omega_3}[a]$, $\omega_2 \in M$, $|M| = \omega_1$ and

(2) $M^\omega \subseteq M$(M is closed under ω-sequence from M).

Let $\pi : M \cong L_\theta[a]$ be the transitive collapsing. Since $|M| = \omega_1, \theta$ is a limit ordinal such that $\omega_1 \leq \theta < \omega_2$. Let j be the inverse of π. Then $j : L_\theta[a] \prec L_{\omega_3}[a]$. Since $\omega_2 \in M$ and $j(\pi(\omega_2)) = \omega_2, \omega_2$ is in the range of j. So $j \restriction \theta \neq id$. Let $\kappa = crit(j)$. Such κ exists and $\kappa < \theta$. Define

$$U = \{X \subseteq \kappa \mid X \in L \wedge \kappa \in j(X)\}.$$

We show that such U is well defined. Since $L_{\omega_3}[a] \models KP$ and $L_\theta[a] \prec L_{\omega_3}[a]$, θ is a-admissible. By assumption θ is an L-cardinal. For any $X \in L$ with $X \subseteq \kappa$, since $\kappa < \theta$, X has L-cardinality less than θ and so $X \in L_\theta$. Hence $X \in L_\theta[a]$ and $j(X)$ is defined.

It is easy to check that U is an L-filter on κ. U is also an L-ultrafilter on κ: if $X \in L, X \subseteq \kappa$ and $\kappa \notin j(X)$, then $\kappa \in j(\kappa) \setminus j(X) = j(\kappa \setminus X)$. Since $\kappa \setminus X \in L$ and $\kappa \setminus X \in L_\theta$, $\kappa \setminus X \in U$. So U is an L-ultrafilter on κ.

From U we can define the ultrapower model L^κ/U where

$$L^\kappa/U = \{[f]_U \mid f : \kappa \to L \text{ and } f \in L\}.$$

Then $L \prec L^\kappa/U$ via the map which sends $x \in L$ to $[c_x]$ where $c_x : \kappa \to \{x\}$ is the constant function with value x.

Claim L^κ/U is well founded.

Proof If not, then there is a sequence $\langle f_n : n \in \omega \rangle$ of constructible functions from κ to L such that for all $n \in \omega, U_n = \{\alpha \in \kappa : f_{n+1}(\alpha) \in f_n(\alpha)\} \in U$. Since $\mathcal{P}(\kappa) \cap L \subseteq L_\theta$, the sequence $\langle U_n : n \in \omega \rangle$ is a sequence of elements of L_θ. Since M is closed under ω-sequence from M, $L_\theta[a]$ is closed under ω-sequence from $L_\theta[a]$. Hence $\langle U_n : n \in \omega \rangle \in L_\theta[a]$. Since $\kappa \in j(U_n)$ for any $n \in \omega$ and j is elementary, we have $\kappa \in j(\bigcap_{n \in \omega} U_n)$. So $\bigcap_{n \in \omega} U_n \neq \emptyset$. Take $\xi \in \bigcap_{n \in \omega} U_n$. Then for any $n \in \omega, f_{n+1}(\xi) \in f_n(\xi)$. Contradiction. $\square$

Fact 2.0.2 *([2]) Suppose $N \models ZF^- + V = L$.*

(a) If N is a transitive proper class, then $N = L$.

(b) If N is a transitive set, then $N = L_{N \cap Ord}$.

Since L^κ/U is well founded, $L \prec L^\kappa/U \cong L$. So we get a non-trivial elementary embedding from L to L, hence $0^\sharp \in L[a]$.[1] $\square$

Theorem 2.0.3 *([18]) $Z_4 +$ **Harrington's** $\star$ implies $0^\sharp$ exists.*

[1]This proof is not a proof in Z_4. This proof takes a submodel of $L_{\omega_3}[a]$. However ω_3 does not exist in Z_4. Also this proof derives the existence of $0^\sharp$ from the existence of a non-trivial elementary embedding from L to L. But the standard proof of "if there exists a non-trivial elementary embedding from L to L, then $0^\sharp$ exists" in [9], [2] and [11] is not a proof in Z_4.

2.1 Forcing background

2.1.1 Almost disjoint forcing

Definition 2.1.1 $\mathcal{F} = \{a_\alpha : \alpha < \lambda\}$ *is an almost disjoint family of size* λ *on* κ *if and only if*

(1) for all $\alpha < \lambda, a_\alpha \subseteq \kappa$ *and* $|a_\alpha| = \kappa$*;*

(2) for all α, β *in* λ *with* $\alpha \neq \beta$*, we have* $|a_\alpha \cap a_\beta| < \kappa$*.*

Suppose κ is a regular cardinal, $\lambda > \kappa$ and $\mathcal{F} = \{a_\alpha : \alpha < \lambda\}$ is an almost disjoint family of size λ on κ. Given $A \subseteq \lambda$, using $\mathcal{F}$ we can force to add a $B \subseteq \kappa$ such that B codes A in the following sense:

$$A = \{\alpha < \lambda : |B \cap a_\alpha| < \kappa\}.$$

Definition 2.1.2 *Given* $\mathcal{F}, A$*, we define the almost disjoint forcing notion* $P_{\mathcal{F},A}$ *as follows:*

$$P_{\mathcal{F},A} = [\kappa]^{<\kappa} \times [A]^{<\kappa}.$$

$$(p, q) \leq (p', q') \leftrightarrow (p \supseteq p' \wedge q \supseteq q' \wedge \forall \alpha \in q'(p \cap a_\alpha \subseteq p')).$$

For any $\alpha \in A$, define

$$D_\alpha = \{(p, q) \in P_{\mathcal{F},A} \mid \alpha \in q\}.$$

For $\alpha < \kappa$ and $\beta \in \lambda \setminus A$, define

$$D_{\alpha,\beta} = \{(p, q) \in P_{\mathcal{F},A} \mid o.t.(p \cap a_\beta) \geq \alpha\}.$$

For $\alpha \in A, D_\alpha$ is dense in $P_{\mathcal{F},A}$ since for any $(p,q) \in P_{\mathcal{F},A}, (p, q \cup \{\alpha\}) \leq (p,q)$ and $(p, q \cup \{\alpha\}) \in D_\alpha$.

Proposition 2.1.3 *For $\alpha < \kappa$ and $\beta \in \lambda \setminus A, D_{\alpha,\beta}$ is dense in $P_{\mathcal{F},A}$.*

Proof Given $(p,q) \in P_{\mathcal{F},A}$, let $S = a_\beta \setminus \bigcup_{\gamma \in q}(a_\beta \cap a_\gamma)$. Since $\beta \notin A$, for $\gamma \in q \subseteq A, |a_\beta \cap a_\gamma| < \kappa$. Since $|q| < \kappa, |S| = \kappa$. Let $p' = p \cup D$ with $D \subseteq S$ and $o.t.(D) = \alpha$. So $o.t.(p' \cap a_\beta) \geq \alpha$. By the definition of S, for all $\gamma \in q, p' \cap a_\gamma \subseteq p$. So $(p', q) \leq (p,q)$ and $(p', q) \in D_{\alpha,\beta}$. □

Proposition 2.1.4 *Let G be $P_{\mathcal{F},A}$-generic over V. Let $B = \bigcup\{p \mid \exists q\,((p,q) \in G)\}$. Then we have*

$$A = \{\alpha < \lambda : |B \cap a_\alpha| < \kappa\}.$$

Proof If $\alpha \in A$, since D_α is dense, there is $(p,q) \in G$ such that $\alpha \in q$. So $B \cap a_\alpha \subseteq p$ and $|B \cap a_\alpha| \leq |p| < \kappa$.

If $\beta \in \lambda \setminus A$, then for all $\alpha < \kappa, D_{\alpha,\beta}$ is dense. So there is $(p,q) \in G$ such that $o.t.(p \cap a_\beta) \geq \alpha$. Since for any $\alpha < \kappa, o.t.(p \cap a_\beta) \geq \alpha$, we have $|B \cap a_\alpha| = \kappa$.

□

Proposition 2.1.5 *$P_{\mathcal{F},A}$ is κ-closed.*

Proof Let $\alpha < \kappa$ and (p_ξ, q_ξ) be a descending sequence of conditions. Let $p' = \bigcup_{\xi<\alpha} p_\xi$ and $q' = \bigcup_{\xi<\alpha} q_\xi$. Note that for $\xi < \alpha$ and $\beta \in q_\xi, p' \cap a_\beta \subseteq p_\xi$. So for any $\xi < \alpha, (p', q') \leq (p_\xi, q_\xi)$. Also since κ is regular, $(p', q') \in P_{\mathcal{F},A}$. □

Proposition 2.1.6 *If $\kappa^{<\kappa} = \kappa$, then $P_{\mathcal{F},A}$ is κ^+-c.c. Especially, if $\kappa = \omega$ or κ is inaccessible, then $P_{\mathcal{F},A}$ is κ^+-c.c.*

Proof Note that for (p, q) and (p, r) in $P_{\mathcal{F},A}$, $(p, q \cup r)$ is a common extension of (p, q) and (p, r). □

Especially, given $A \subseteq \omega_1$ and an almost disjoint family $\mathcal{F} = \{x_\alpha \mid \alpha < \omega_1\}$ on ω, we can code A by a real x via forcing over $P_{\mathcal{F},A}$.

Define $(p, q) \in P_{\mathcal{F},A}$ if and only if

(i) p is a finite subset of ω,

(ii) q is a finite subset of A and

(iii) $(p, q) \leq (p', q') \leftrightarrow (p \supseteq p' \wedge q \supseteq q' \wedge \forall\alpha \in q'(p \cap x_\alpha \subseteq p'))$.

If G is $P_{\mathcal{F},A}$-generic over V, then we have

$$A = \{\alpha < \omega_1 : |x \cap a_\alpha| < \omega\}$$

where

$$x = \bigcup\{p \mid \exists q\, ((p, q) \in G)\}.$$

Example 2.1.7 *For any $n \in \omega$ and any countable transitive model M of $ZFC + V = L$, we can force over M to get a real x such that in $M[x]$, ω_{n+1}^M is collapsed to ω_n^M, all cardinals $\leq \omega_n^M$ are preserved and all cardinals $\geq \omega_{n+2}^M$ are preserved.*

Proof Force $Fn(\omega_n, \omega_{n+1}, \omega_n)$ over M to collapse ω_{n+1} to ω_n. Let $A_0 \subseteq \omega_n$ code the collapsing function. Since $Fn(\omega_n, \omega_{n+1}, \omega_n)$ is ω_n-closed and ω_{n+2}-c.c, in $M[A_0]$, all cardinals $\leq \omega_n^M$ are preserved and all cardinals $\geq \omega_{n+2}^M$ are preserved. In $M[A_0]$, take an almost disjoint family $\mathcal{F}_0$ of size ω_n on ω_{n-1}. In $M[A_0]$, force over $P_{\mathcal{F}_0, A_0}$ by almost disjoint forcing to get $A_1 \subseteq \omega_{n-1}$ which codes A_0 such that $A_0 \in M[A_1]$. Note that $P_{\mathcal{F}_0, A_0}$ is ω_{n-1}-closed and ω_n-c.c. So $M[A_1]$ and $M[A_0]$ have the same cardinals. Continue this process. In In $M[A_m]$, take an almost disjoint family $\mathcal{F}_m$ of size ω_{n-m} on ω_{n-1-m}. In $M[A_m]$, force over $P_{\mathcal{F}_m, A_m}$ by almost disjoint forcing to get $A_{m+1} \subseteq \omega_{n-1-m}$ which codes A_m such that $A_m \in M[A_{m+1}]$. Note that $P_{\mathcal{F}_m, A_m}$ is ω_{n-1-m}-closed and ω_{n-m}-c.c. So $M[A_m]$ and $M[A_{m+1}]$ have the same cardinals. Finally, when $m = n - 1$ we get a real x such that in $M[x]$,

(i) for all $0 \leq k \leq n - 1$, x codes A_k and $A_k \in M[x]$;

(ii) ω_{n+1}^M is collapsed to ω_n^M;

(iii) all cardinals $\leq \omega_n^M$ are preserved and all cardinals $\geq \omega_{n+2}^M$ are preserved.

□

2.1.2 Some notions of forcing

Levy collapse

Definition 2.1.8 *Suppose γ is a regular cardinal and $\kappa > \gamma$.*

(i) $Col(\gamma, \kappa) = \{p \mid p : \gamma \to \kappa$ *and* $|dom(p)| < \gamma\}$.

(ii) $Col(\gamma, < \kappa) = \{p \mid$ *p is a function,* $|dom(p)| < \gamma, dom(p) \subseteq \kappa \times \gamma$ *and for any* $(\alpha, \beta) \in dom(p), p(\alpha, \beta) < \alpha\}$.

(iii) $Fn(I, J, \kappa) = \{p \mid p : I \to J$ *and* $|dom(p)| < \kappa\}$.[2]

In all these forcing notions, $p \leq q$ *if and only if* $p \supseteq q$.

Fact 2.1.9 *([9], [12])*

(i) $Col(\gamma, \kappa)$ *collapses* κ *to* γ. $Col(\gamma, < \kappa)$ *collapses any* $\gamma < \lambda < \kappa$ *to* γ *and collapses* κ *to* γ^+.

(ii) $Col(\gamma, \kappa), Col(\gamma, < \kappa)$ *are* γ*-closed.*

(iii) $Fn(I, J, \kappa)$ *is* $(|J|^{<\kappa})^+$*-c.c. If* κ *is regular, then* $Fn(I, J, \kappa)$ *is* κ*-closed. Especially, if* $\kappa^{<\gamma} = \kappa$*, then* $Col(\gamma, \kappa)$ *is* κ^+*-c.c.*

(iv) If κ *is regular and for any* $\alpha < \kappa, \alpha^{<\gamma} < \kappa$*, then* $Col(\gamma, < \kappa)$ *is* κ*-c.c.*

Theorem 2.1.10 *(Levy,[9])* *Suppose* κ *is a regular cardinal,* $\lambda > \kappa$ *is an inaccessible cardinal and* G *is* $Col(\kappa, < \lambda)$*-generic over* V.

(a) Every α *such that* $\kappa \leq \alpha < \lambda$ *has cardinality* κ *in* $V[G]$.

(b) Every cardinal $\leq \kappa$ *and every cardinal* $\geq \lambda$ *remains a cardinal in* $V[G]$.

[2]Note that $Col(\gamma, \kappa) = Fn(\gamma, \kappa, \gamma)$.

Hence $V[G] \models \lambda = \kappa^+$.

Proof By Fact 2.1.9 and 1.1.9, $Col(\kappa, < \lambda)$ is λ-c.c. and κ-closed. $\square$

Shooting a club

If $V[G]$ is a generic extension, then every club $C \in V$ on ω_1 remains a club in $V[G]$, provided that ω_1 is preserved. But a stationary set S in V may no longer be stationary in $V[G]$ since there may be a club $C \in V[G]$ disjoint from S.

Theorem 2.1.11 *(Baumgartner, Harrington and Kleinberg, [6]) Let* $S \subseteq \omega_1$ *be stationary. Then there exists an* ω_1*-preserving generic extension* $V[G]$ *such that* $V[G] \models \exists C \subseteq S(C$ *is a club on* $\omega_1)$.

Proof Let $P_S = \{p : p$ is a closed bounded subset of ω_1 and $p \subseteq S\}$. For $p, q \in P_S, p \leq q$ if and only if p end extends q. i.e. $p \supseteq q$ and for any $\alpha \in p \setminus q, \alpha > sup(q)$.

Lemma 2.1.12 *If* G *is* P_S*-generic over* V*, then* $V[G] \models \bigcup G$ *is a club on* ω_1.

Proof $\bigcup G$ is unbounded in ω_1: Fix $\alpha < \omega_1$. $D_\alpha = \{p \in P_S \mid sup(p) > \alpha\}$ is dense in P_S. (Given $q \in P_S$, since S is stationary and hence unbounded in ω_1, $\exists \beta \in S(\beta > sup(q) \wedge \beta > \alpha)$. So $q \cup \{\beta\}$ end extends q and $q \cup \{\beta\} \in D_\alpha$.)

So there is $p \in G$ such that $sup(p) > \alpha$. Since p is closed, $sup(p) \in p$ and hence $sup(p) \in \bigcup G$.

$\bigcup G$ is closed: Suppose α is a limit point of $\bigcup G$. Since $\alpha < \omega_1$, there is $\beta \in \bigcup G$ such that $\beta > \alpha$. Let $\beta \in q \in G$. So there is $q \in G$ such that $\alpha \in \cup q = sup(q)$.

$\alpha \cap \bigcup G \subseteq q$: Let $\gamma \in \alpha \cap \bigcup G$ and $\gamma \in q' \in G$. If q end extends q', then $\gamma \in q$. If q' end extends q, then $\gamma \in q$ also holds. (If $\gamma \notin q$, then $\gamma > sup(q)$ since $\gamma \in q' \setminus q$. But $\gamma < \alpha < sup(q)$. Contradiction.)

Since α is a limit point of $\bigcup G, \alpha \cap \bigcup G \subseteq q$ and q is closed, we have $\alpha \in q$ and hence $\alpha \in \bigcup G$. □

So we have shown that $\bigcup G$ is a club on $\omega_1^{V[G]}$. Now we show that ω_1 is preserved and hence $\omega_1^{V[G]} = \omega_1^V$.

Lemma 2.1.13 *P_S is ω_1-distributive*[3] *and hence preserves ω_1.*

Proof It suffices to show that any countable set of ordinals in $V[G]$ is in the ground model. Let $p \Vdash \dot{f} : \omega \to Ord$. It suffices to show that there exist $q \le p$ and g such that $q \Vdash \dot{f} = \check{g}$.

By induction on α we construct a chain $\{A_\alpha \mid \alpha < \omega_1\}$ of countable subsets

[3]The usually used term for "ω_1-distributive" is "ω-distributive". Our use is different from the common usage but is consistent with Definition 1.1.8. We explain this point here for the throughout use of "ω_1-distributive" in this thesis.

of P_S. Let $A_0 = \{p\}$. If α is a limit ordinal, $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$. Given A_α, let $\gamma_\alpha = sup(\{sup(q) : q \in A_\alpha\})$. Since $A_\alpha \subseteq P_S$ is countable, $\gamma_\alpha < \omega_1$.

For each $q \in A_\alpha$ and each $n \in \omega$, choose some $r = r(q,n) \in P_S$ such that $r \leq q, r$ decides $\dot{f}(n)$ and $sup(r) > \gamma_\alpha$.[4] Let $A_{\alpha+1} = A_\alpha \cup \{r(q,n) : q \in A_\alpha, n \in \omega\}$.

The sequence $\{\gamma_\alpha : \alpha < \omega_1\}$ is increasing and continuous. Let

$$C = \{\lambda < \omega_1 : \alpha < \lambda \to \gamma_\alpha < \lambda\}.$$

Since C is a club on ω_1 and S is stationary, there exists a limit ordinal $\lambda \in C \cap S$. Let $\{\alpha_n : n \in \omega\}$ be an increasing sequence with limit λ. By the definition of C, $\lim_n \gamma_{\alpha_n} = \lambda$.

From the construction of $\{A_\alpha : \alpha < \omega_1\}$, there is a sequence $\{p_n : n \in \omega\}$ such that $p_0 = p$ and for any $n \in \omega$,

(1) $p_{n+1} \in A_{\alpha_n+1}$,

(2) $p_{n+1} \leq p_n$,

(3) $\gamma_{\alpha_n} < sup(p_{n+1})$ and p_{n+1} decides $\dot{f}(n)$.

Since $sup(p_{n+1}) \leq \gamma_{\alpha_n+1}$, we have $\lim_{n\in\omega} sup(p_n) = \lim_n \gamma_{\alpha_n} = \lambda$. Since $\lambda \in S, q = \bigcup_{n\in\omega} p_n \cup \{\lambda\}$ is closed and $q \subseteq S$. Hence $q \in P_S$. Since $q \leq p_n$ for all $n \in \omega, q$ decides each $\dot{f}(n)$. For $n \in \omega$, let $g(n)$ be the value of $\dot{f}(n)$ decided by q. Then we have $q \Vdash \dot{f} = \check{g}$. $\square$

[4] r decides $\dot{f}(n)$ means for some $x \in V, r \Vdash \dot{f}(n) = \check{x}$.

□

Remark For stationary $S \subseteq \omega_1$, if $\omega_1 \setminus S$ is also stationary, then the stationarity of $\omega_1 \setminus S$ is destroyed by P_S since $\omega_1 \setminus S$ is disjoint from the new added club $C \subseteq S$ in $V[G]$.

As a summary, P_S has the following properties:

(i) P_S is not proper.

(ii) P_S is not countably closed.

(iii) P_S is ω_1-distributive and adds no new reals.

(iv) $|P_S| = 2^\omega$. Assuming CH, P_S preserves all cardinals.[5]

(v) Suppose G is P_S-generic over V and $C \subseteq S$ is the new club in $V[G]$. For all $\alpha < \omega_1, C \cap \alpha \in V$.

[5] If $2^\omega > \omega_1$, then 2^ω is collapsed to ω_1 in $V[G]$ where G is P_S-generic over V.

2.2 Z_2+ Harrington's $\star$ does not imply $0^\sharp$ exists

In this section[6], we prove that $Z_2 +$ **Harrington's** $\star$ does not imply $0^\sharp$ exists.[7]

This answers Question 1.2.4.

Theorem 2.2.1 *(R.B.Jensen and R.M.Solovay, [10]) Let M be a countable transitive model of $ZFC + V = L$. Let κ be an inaccessible cardinal in M. Then there is a real x such that*

$$M[x] \models \text{“}ZFC \wedge \omega_1 = \kappa^M\text{”}.$$

Moreover, it preservers cardinals $\geq \kappa$.

Theorem 2.2.2 *Z_2+ **Harrington's** $\star$ does not imply $0^\sharp$ exists.*[8]

Proof We want to prove that $Z_2 +$ **Harrington's** $\star$ does not imply "$0^\sharp$ exists". We assume that $Z_2 +$ **Harrington's** $\star$ is consistent and suppose

[6] We examine whether $Z_3 +$ **Harrington's** $\star$ implies $0^\sharp$ exists only after we get the negative result in this section. The solution for question 1.2.5 partly arises from the solution for question 1.2.4. Our answer for question 1.2.4 in this section provides a motivation for the proof in Section 2.3.

[7] In the context of Z_2, **Harrington's** $\star$ is equivalent to the statement "$\exists x \in 2^\omega \forall \alpha(\alpha$ is x-admissible $\rightarrow \alpha$ is an L-cardinal)". We make a convention that whenever we talk about $Z_2 +$ **Harrington's** $\star$, **Harrington's** $\star$ denotes the statement "$\exists x \in 2^\omega \forall \alpha(\alpha$ is x-admissible $\rightarrow \alpha$ is an L-cardinal)".

[8] Motivation of the proof: We work in a minimal model of "$Z_2 + 0^\sharp$ exists". By the use of Jensen-Solovay's theorem, we get a real x_0 such that in $L[x_0], C = \{\eta < \omega_1 \mid \eta$ is an L-cardinal and $L_\eta[x_0] \prec L_{\omega_1}[x_0]\}$ is a club on ω_1. Do almost disjoint forcing over $L[x_0]$ to get a real x_1. Let $x = x_0 \oplus x_1$ and θ be the least ordinal such that $L_\theta[x] \models Z_2$. We show that $L_\theta[x] \models$ **Harrington's** $\star$ and in fact x is the witness real for **Harrington's** $\star$. To show this, it suffices to show that if $\alpha < \theta$ and α is x-admissible, then α is an L-cardinal. Define $\gamma_0 = \sup(\{\gamma < \alpha \mid (L_\gamma[x_0, C], C \cap \gamma) \prec (L_{\omega_1}[x_0, C], C)\})$. To show α is an L-cardinal it suffices to show that $\gamma_0 = \alpha$. This depends on the clever design of almost disjoint forcing and the use of x-admissibility of α.

$(M,E) \models Z_2+$ **Harrington's** $\star$. If $(M,E) \models$ "$0^\sharp$ does not exist", we are done. Let us assume that $(M,E) \models$ "$0^\sharp$ exists". We show that "Z_2+ **Harrington's** $\star$ $+0^\sharp$ does not exist" is consistent. We work in (M,E). Since $0^\sharp$ exists, $\exists x \in \omega^\omega \, \exists \alpha (L_\alpha[x] \models Z_2 + 0^\sharp$ exists). Suppose δ^* is the least ordinal such that $\exists x \in \omega^\omega (L_{\delta^*}[x] \models Z_2 + 0^\sharp$ exists). Fix such least δ^* and some real z^* such that $L_{\delta^*}[z^*] \models Z_2 + 0^\sharp$ exists. Note that $\delta^* < \omega_1$ and in fact $\delta^* < \omega_1^{L[z^*]}$ since δ^* defined in V is the same as defined in $L[z^*]$. Now we work in $L_{\delta^*}[z^*]$ in which "$Z_2 + 0^\sharp$ exists" holds.

Goal Find a real x such that $L_\alpha[x] \models$ "Z_2+ **Harrington's** $\star + 0^\sharp$ does not exist" for some ordinal $\alpha < \delta^*$.

We find such a real x by forcing over L to get x such that $L_\alpha[x] \models$ "Z_2+ **Harrington's** $\star + 0^\sharp$ does not exist" for some ordinal $\alpha < \delta^*$.

Convention Throughout the following proof, we work in $L_{\delta^*}[z^*]$ in which "$Z_2 + 0^\sharp$ exists" holds.

Take the least inaccessible κ in L. By Jensen-Solovay's theorem, we get a real x_0 such that $L[x_0] \models \kappa^L = \omega_1$. In $L, \{\alpha < \kappa \mid \alpha$ is an L-cardinal$\}$ is a club on κ. In $L[x_0], \{\alpha < \omega_1 \mid \alpha$ is an L-cardinal$\}$ is a club on ω_1.

In the following we assume that $V = L[x_0]$. Let $C = \{\eta < \omega_1 \mid \eta$ is an L-cardinal and $L_\eta[x_0] \prec L_{\omega_1}[x_0]\}$. Note that C is a club on ω_1 and for each $\eta \in C$, any $\xi < \eta$ is countable in $L_\eta[x_0]$.

Define $F : \omega^\omega \to \omega^\omega$ as follows: if $y \subseteq \omega$ codes γ, then $F(y)$ is a real which codes $(\beta, C \cap \beta)$ where β is the least element of C such that $\beta > \gamma$ and $(L_\beta[x_0, C], C \cap \beta) \prec (L_{\omega_1}[x_0, C], C)$. Since C is a club on ω_1, in the definition of $F(y)$ such β exists.

To define an almost disjoint sequence $\langle \delta_\beta \mid \beta < \omega_1 \rangle$, we firstly define a sequence of distinct reals $\langle \sigma_\beta \mid \beta < \omega_1 \rangle$. Let σ_0 be the $<_{L[x_0,C]}$-least real. Fix $\gamma < \omega_1$. Suppose we have defined $\langle \sigma_\beta \mid \beta < \gamma \rangle$. Since γ is countable and $L[x_0, C] \models 2^\omega = \omega_1$, let σ_γ be the $<_{L[x_0,C]}$-least real which is different from σ_β for any $\beta < \gamma$. From $\langle \sigma_\beta \mid \beta < \omega_1 \rangle$, we can define an almost disjoint sequence $\langle \delta_\beta \mid \beta < \omega_1 \rangle$ as follows. Let $\langle s_i \mid i \in \omega \rangle$ be an injective and recursive enumeration of $\omega^{<\omega}$. For any $\beta < \omega_1$, define

$$\delta_\beta = \{ i \in \omega \mid \exists m \in \omega (s_i = \sigma_\beta \cap m) \}.$$

It is easy to check that $\langle \delta_\beta : \beta < \omega_1 \rangle$ is a sequence of almost disjoint reals. Note that $\langle \delta_i : i < \omega \rangle$ is recursive.

Let $\langle x_\alpha \mid \alpha < \omega_1 \rangle$ be the enumeration of $\mathcal{P}(\omega)$ in $L[x_0, C]$ in the order of construction. Define $Z_F \subseteq \omega_1$ as follows.

$$Z_F = \{ \alpha \cdot \omega + i \mid \alpha < \omega_1 \wedge i \in F(x_\alpha) \}.$$

Now we do almost disjoint forcing to code Z_F via $\langle \delta_\beta \mid \beta < \omega_1 \rangle$. Then we get a new real x_1 such that $\alpha \in Z_F \Leftrightarrow |x_1 \cap \delta_\alpha| < \omega$. The almost disjoint forcing preserves all cardinals since it is *c.c.c.* Let $x = x_0 \oplus x_1$.

Now we work in $L[x]$. Take the least θ such that $L_\theta[x] \models Z_2$. Such θ exists and $\theta < \omega_1$. We show that $L_\theta[x] \models$ **Harrington's** $\star$. It suffices to show that if $\alpha < \theta$ is x-admissible, then α is an L-cardinal. Suppose $\alpha < \theta$ is x-admissible. We show that α is an L-cardinal.

Define

$$\gamma_0 = \sup(\{\gamma < \alpha \mid (L_\gamma[x_0, C], C \cap \gamma) \prec (L_{\omega_1}[x_0, C], C)\}).$$

If there is no $\gamma < \alpha$ such that $(L_\gamma[x_0, C], C \cap \gamma) \prec (L_{\omega_1}[x_0, C], C)$, then let $\gamma_0 = 0$. From the definition of γ_0, we have $\gamma_0 \leq \alpha$. Note that if $\gamma < \omega_1$ and $(L_\gamma[x_0, C], C \cap \gamma) \prec (L_{\omega_1}[x_0, C], C)$, then $\gamma \in C$. So if $\gamma_0 > 0$, then $\gamma_0 \in C$, $(L_{\gamma_0}[x_0, C], C \cap \gamma_0) \prec (L_{\omega_1}[x_0, C], C)$ and hence $L_{\gamma_0}[x_0] = L_{\gamma_0}[x_0, C]$.

We assume that $\gamma_0 < \alpha$. Let α_0 be the least x_0-admissible ordinal such that $\alpha_0 > \gamma_0$. Since α is x_0-admissible and $\gamma_0 < \alpha$, we have $\alpha_0 \leq \alpha$.

Claim

$$C \cap \alpha_0 = C \cap (\gamma_0 + 1).$$

Proof We show that $C \cap \alpha_0 \subseteq C \cap (\gamma_0 + 1)$. Suppose $\gamma \in C \cap \alpha_0$ and $\gamma > \gamma_0$. Since $\gamma \in C$, $L_\gamma[x_0] \prec L_{\omega_1}[x_0]$. Since α_0 is definable from γ_0 and x_0, we have α_0 is definable in $L_\gamma[x_0]$. So $\alpha_0 \leq \gamma$. Contradiction. $\square$

Since $C \cap \alpha_0 = C \cap (\gamma_0 + 1)$, we have $L_{\alpha_0}[C, x_0] = L_{\alpha_0}[C \cap \gamma_0, x_0]$.

Claim γ_0 is countable in $L_{\alpha_0}[C \cap \gamma_0, x_0]$.

Proof Suppose γ_0 is not countable in $L_{\alpha_0}[C \cap \gamma_0, x_0]$. Let P be the partial order for almost disjoint coding Z_F via the almost disjoint system $\langle \delta_\beta \mid \beta < \omega_1 \rangle$. Note that P is definable over $L_{\omega_1+1}[x_0, C]$. Since $(L_{\gamma_0}[x_0, C], C \cap \gamma_0) \prec (L_{\omega_1}[x_0, C], C)$, we have $P \cap L_{\gamma_0}[x_0, C]$ is definable over $L_{\gamma_0+1}[x_0, C]$. Let $P^* = P \cap L_{\gamma_0}[x_0, C]$. Note that $P^* \in L_{\alpha_0}[C \cap \gamma_0, x_0]$. Since P^* is c.c.c in $L_{\alpha_0}[x_0, C \cap \gamma_0]$, we have $\gamma_0 = \omega_1^{L_{\alpha_0}[C \cap \gamma_0, x_0]}$.

We show that x_1 is generic over $L_{\alpha_0}[C \cap \gamma_0, x_0]$ for P^*. Let $Y \subseteq P^*$ be a maximal antichain with $Y \in L_{\alpha_0}[C \cap \gamma_0, x_0]$. Since P^* is c.c.c in $L_{\alpha_0}[C \cap \gamma_0, x_0]$ and $\gamma_0 = \omega_1^{L_{\alpha_0}[C \cap \gamma_0, x_0]}$, we have $Y \in L_{\gamma_0}[C \cap \gamma_0, x_0] = L_{\gamma_0}[C, x_0] = L_{\gamma_0}[x_0]$. Since $(L_{\gamma_0}[x_0, C], C \cap \gamma_0) \prec (L_{\omega_1}[x_0, C], C)$, it follows that Y is a maximal antichain in P. So the filter given by x_1 meets Y and hence x_1 is generic over $L_{\alpha_0}[C \cap \gamma_0, x_0]$ for P^*.

Note that $L_{\alpha_0}[x_0, C \cap \gamma_0] \cap \omega^\omega = L_{\gamma_0}[x_0, C \cap \gamma_0] \cap \omega^\omega$. So $\gamma_0 = \omega_1^{L_{\alpha_0}[C \cap \gamma_0, x_0]} = \omega_1^{L_{\alpha_0}[C \cap \gamma_0, x_0][x_1]} = \omega_1^{L_{\gamma_0}[C \cap \gamma_0, x_0][x_1]}$. Since $L_{\gamma_0}[x_0, C \cap \gamma_0][x_1] = L_{\gamma_0}[x]$, we have $\gamma_0 = \omega_1^{L_{\gamma_0}[x]}$ and so $L_{\gamma_0}[x] \models Z_2$ which contradicts that $\gamma_0 < \alpha_0 < \theta$ and θ is the least ordinal such that $L_\theta[x] \models Z_2$. $\square$

Note that for any $\eta < \alpha_0, \eta$ is countable in $L_{\alpha_0}[C, x_0] = L_{\alpha_0}[C \cap \gamma_0, x_0]$. From our definition of $\langle \delta_\beta : \beta < \omega_1 \rangle$ and $\langle x_\alpha \mid \alpha < \omega_1 \rangle$, we have:

(i) For each $\eta < \alpha_0, \langle \delta_\beta : \beta < \eta \rangle \in L_{\alpha_0}[C, x_0] = L_{\alpha_0}[C \cap \gamma_0, x_0]$.

(ii) For each $\eta < \alpha_0, \langle x_\beta : \beta < \eta \rangle \in L_{\alpha_0}[C, x_0] = L_{\alpha_0}[C \cap \gamma_0, x_0]$.

(iii) $\langle x_\alpha \mid \alpha < \alpha_0 \rangle$ enumerates $\mathcal{P}(\omega) \cap L_{\alpha_0}[C \cap \gamma_0, x_0]$.

Claim

$$C \cap \gamma_0 \in L_{\gamma_0+1}[x].$$

Proof If $\gamma_0 = 0$, this is trivial. Suppose $\gamma_0 > 0$. Let

$$D = \{\gamma \in C \cap \theta \mid (L_\gamma[x_0, C], C \cap \gamma) \prec (L_{\omega_1}[x_0, C], C)\}.$$

We prove by induction that for any $\gamma \in D, C \cap \gamma$ is definable in $L_\gamma[x]$ from x. Fix $\gamma \in D$. Suppose for any $\gamma' \in D \cap \gamma$, $C \cap \gamma' \in L_{\gamma'+1}[x]$. We show that $C \cap \gamma$ is definable in $L_\gamma[x]$ from x.

Let η be the least ordinal such that $L_\eta[x_0, C \cap \gamma]$ is admissible. Since $\gamma \in \theta, \gamma$ is countable in $L_\eta[x_0, C \cap \gamma]$. (If γ is not countable in $L_\eta[x_0, C \cap \gamma]$, then by the similar argument as we show that γ_0 is countable in $L_{\alpha_0}[C \cap \gamma_0, x_0]$, x_1 is generic over $L_\eta[x_0, C \cap \gamma]$ and so $L_\gamma[x] \models Z_2$ which leads a contradiction).

Since $(L_\gamma[x_0, C], C \cap \gamma) \prec (L_{\omega_1}[x_0, C], C)$, we have $L_\gamma[x_0] = L_\gamma[x_0, C]$. Note that $C \cap \eta = C \cap (\gamma + 1)$. Note that for any $\beta < \eta, \beta$ is countable in $L_\eta[x_0, C \cap \gamma]$. From our definitions, for any $\beta < \eta$ we have:

(i) $\langle x_\xi \mid \xi \in \beta \rangle \in L_\eta[x_0, C \cap \gamma]$.

(ii) $\langle \delta_\xi \mid \xi \in \beta \rangle \in L_\eta[x_0, C \cap \gamma]$.

(iii) $\langle x_\xi \mid \xi \in \eta \rangle$ enumerates $\mathcal{P}(\omega) \cap L_\eta[x_0, C] = \mathcal{P}(\omega) \cap L_\eta[x_0, C \cap \gamma]$.

Suppose $y \subseteq \omega$ and $y \in L_\eta[x_0, C \cap \gamma]$. Then $y = t_\xi$ for some $\xi < \eta$. Note that $\xi \cdot \omega + i < \eta$ and $i \in F(y)$ if and only if $|x_1 \cap \delta_{\xi \cdot \omega + i}| < \omega$. So $F(y) \in L_\eta[x_0, C \cap \gamma][x_1]$. Hence we have shown that if $y \in \mathcal{P}(\omega) \cap L_\eta[x_0, C \cap \gamma]$, then $F(y) \in L_\eta[x, C \cap \gamma]$.

Case 1: There exists β such that γ is the least element of C such that $\gamma > \beta$ and $(L_\gamma[x_0, C], C \cap \gamma) \prec (L_{\omega_1}[x_0, C], C)$. Since γ is countable in $L_\eta[x_0, C \cap \gamma]$, we have β is countable in $L_\eta[x_0, C \cap \gamma]$. Take a real $y \in L_\eta[x_0, C \cap \gamma]$ such that y codes β. So $F(y)$ is a real which codes $(\gamma, C \cap \gamma)$ and $F(y) \in L_\eta[x, C \cap \gamma]$. Since $\gamma \in D$ and η is the least ordinal such that $L_\eta[x_0, C \cap \gamma]$ is admissible, we have $F(y)$ is definable in $L_\gamma[x_0, C \cap \gamma][x] = L_\gamma[x]$ from x. Since $F(y)$ codes $C \cap \gamma$, we have $C \cap \gamma$ is definable in $L_\gamma[x]$ from x. So $C \cap \gamma \in L_{\gamma+1}[x]$.

Case 2: Such β does not exist. Then γ is a limit point of $\{\gamma' \in C \mid (L_{\gamma'}[x_0, C], C \cap \gamma') \prec (L_{\omega_1}[x_0, C], C)\}$. Let $\gamma = \sup(\{\gamma_n : n \in \omega\})$ where $\gamma_n \in D$. So $C \cap \gamma_n \in L_{\gamma_n+1}[x]$ for any $n \in \omega$. Note that $C \cap \gamma = \bigcup_{n \in \omega}(C \cap \gamma_n)$. So $C \cap \gamma \in L_{\gamma+1}[x]$.

Since $\gamma_0 \in D$, we have $C \cap \gamma_0 \in L_{\gamma_0+1}[x]$. $\square$

Claim If $y \subseteq \omega$ and $y \in L_{\alpha_0}[C \cap \gamma_0, x_0]$, then $F(y) \in L_{\alpha_0}[x]$.

Proof Since $\langle x_\alpha \mid \alpha < \alpha_0 \rangle$ enumerates $\mathcal{P}(\omega) \cap L_{\alpha_0}[C \cap \gamma_0, x_0]$, we have $y = x_\xi$ for some $\xi < \alpha_0$. Note that for any $\xi < \alpha_0, \xi \cdot \omega + i < \alpha_0$ for any $i \in \omega$. By the definition of $Z_F, i \in F(y) \Leftrightarrow i \in F(x_\xi) \Leftrightarrow \xi \cdot \omega + i \in Z_F \Leftrightarrow |x_1 \cap \delta_{\xi\cdot\omega+i}| < \omega$. Since for each $\eta < \alpha_0$, $\langle \delta_\beta : \beta < \eta \rangle \in L_{\alpha_0}[C \cap \gamma_0, x_0]$ and $\xi \cdot \omega + i < \alpha_0$, we have $F(y) \in L_{\alpha_0}[C \cap \gamma_0, x_0][x_1]$. Since $C \cap \gamma_0 \in L_{\gamma_0+1}[x]$, we have $L_{\alpha_0}[C \cap \gamma_0, x_0] \subseteq L_{\alpha_0}[x]$. So $F(y) \in L_{\alpha_0}[x]$. $\square$

Since γ_0 is countable in $L_{\alpha_0}[C \cap \gamma_0, x_0]$, there exists a real $y \in L_{\alpha_0}[C \cap \gamma_0, x_0]$ such that y codes γ_0. Note that $F(y)$ codes $(\gamma_1, C \cap \gamma_1)$ where γ_1 is the least element of C such that $\gamma_1 > \gamma_0$ and $(L_{\gamma_1}[x_0, C], C \cap \gamma_1) \prec (L_{\omega_1}[x_0, C], C)$.

Since $F(y)$ codes $(\gamma_1, C \cap \gamma_1)$ and $F(y) \in L_{\alpha_0}[x]$, we have $\gamma_0 < \gamma_1 < \alpha_0 \leq \alpha$. Since $\gamma_1 < \alpha$ and $(L_{\gamma_1}[x_0, C], C \cap \gamma_1) \prec (L_{\omega_1}[x_0, C], C)$, by the definition of γ_0 we have $\gamma_1 \leq \gamma_0$. Contradiction.

So the assumption that $\gamma_0 < \alpha$ is false. Hence $\gamma_0 = \alpha$. So $(L_\alpha[x_0, C], C \cap \alpha) \prec (L_{\omega_1}[x_0, C], C)$. Hence $\alpha \in C$ and α is an L-cardinal.

We have shown that $L_\theta[x] \models$ **Harrington's** $\star$. Now we show that $L_\theta[x] \models$ "$0^\sharp$ does not exist". Suppose not. i.e. $L_\theta[x] \models 0^\sharp$ exists. Since $\theta < \delta^*$, this contradicts the fact that δ^* is the least ordinal such that $\exists x \in \omega^\omega (L_{\delta^*}[x] \models Z_2 + 0^\sharp \text{ exists})$.

So we get a model $L_\theta[x]$ such that

$$L_\theta[x] \models \textbf{Harrington's } \star \; + Z_2 + 0^\sharp \text{ does not exist.}$$

Hence Z_2 +**Harrington's** $\star \nvdash 0^\sharp$ exists. $\square$

Remark In this proof, we only need to take the least weakly inaccessible cardinal κ in L.

2.3 Z_3+ Harrington's $\star$ does not imply $0^\sharp$ exists

In this section, we prove that Z_3+ **Harrington's** $\star$ does not imply $0^\sharp$ exists. This answers Question 1.2.5.

2.3.1 Weakly reflecting property and strong reflecting property

Definition 2.3.1 *Let $\gamma \geq \omega_1$ be an L-cardinal.*

(i) γ has strong reflecting property if and only if for some uncountable regular cardinal $\kappa > \gamma$,

$$\forall X((X \prec H_\kappa \wedge |X| = \omega \wedge \gamma \in X) \rightarrow \bar{\gamma} \textit{ is an L-cardinal})$$

where $\bar{\gamma}$ is the image of γ under the transitive collapse of X.

(ii) γ has weakly reflecting property if and only if for some uncountable regular cardinal $\kappa > \gamma$,

$$\exists X(X \prec H_\kappa \wedge |X| = \omega \wedge \gamma \in X \wedge \bar{\gamma} \textit{ is an L-cardinal})$$

where $\bar{\gamma}$ is the image of γ under the transitive collapse of X.

Fact 2.3.2 *([8]) For uncountable regular cardinal κ, $\{X \mid |X| = \omega \wedge X \prec H_\kappa\}$ is a closed and unbounded subset of $[H_\kappa]^\omega$.*

Proposition 2.3.3 *Suppose $\gamma \geq \omega_1$ is an L-cardinal. Then the following are equivalent:*

(1) γ has strong reflecting property.

(2) For any uncountable regular cardinal $\kappa > \gamma$,

$$\forall X((X \prec H_\kappa \wedge |X| = \omega \wedge \gamma \in X) \rightarrow \bar{\gamma} \text{ is an } L\text{-cardinal})$$

where $\bar{\gamma}$ is the image of γ under the transitive collapse of X.

(3) For some uncountable regular cardinal $\kappa > \gamma$, $\{X \subseteq H_\kappa \mid$ if $X \prec H_\kappa, |X| = \omega$ and $\gamma \in X$, then $\bar{\gamma}$ is an L-cardinal$\}$ contains a club subset of $[H_\kappa]^\omega$ where $\bar{\gamma}$ is the image of γ under the transitive collapse of X.

(4) There exists $F : \gamma^{<\omega} \rightarrow \gamma$ such that for any $X \subseteq \gamma$, if X is countable and $F``X^{<\omega} \subseteq X$,[9] then o.t.$(X)$ is an L-cardinal.

(5) For any uncountable regular cardinal $\kappa > \gamma$, $\{X \subseteq H_\kappa \mid$ if $X \prec H_\kappa, |X| = \omega$ and $\gamma \in X$, then $\bar{\gamma}$ is an L-cardinal$\}$ contains a club subset of $[H_\kappa]^\omega$ where $\bar{\gamma}$ is the image of γ under the transitive collapse of X.

[9] If $F``X^{<\omega} \subseteq X$, we say that X is closed under F.

Proof Note that $(2) \Rightarrow (1), (1) \Rightarrow (3), (2) \Rightarrow (5)$ and $(5) \Rightarrow (3)$. It suffices to show that $(4) \Rightarrow (2)$ and $(3) \Rightarrow (4)$.

$(4) \Rightarrow (2)$ Let $F : \gamma^{<\omega} \to \gamma$ be such that for any $Z \subseteq \gamma$, if Z is countable and $F``Z^{<\omega} \subseteq Z$, then $o.t.(Z)$ is an L-cardinal. Suppose $\kappa > \gamma$ is regular, $X \prec H_\kappa$, $|X| = \omega$ and $\gamma \in X$. We show that $\bar{\gamma}$ is an L-cardinal. Since $\kappa > \gamma, \mathcal{P}(\gamma) \subseteq H_\kappa$. So $F : \gamma^{<\omega} \to \gamma$ is in H_κ. Note that for $\kappa \geq \omega_1, \{\alpha < \omega_1 \mid \alpha$ is an L-cardinal $\}$ is definable in H_κ. Since $\gamma \in X$, the property of F is definable in H_κ. That is there exists a formula $\varphi(x, y)$ such that $\{F \in H_\kappa \mid H_\kappa \models \varphi[F, \gamma]\} = \{F \mid F : \gamma^{<\omega} \to \gamma$ and for any $Z \subseteq \gamma$, if Z is countable and $F``Z^{<\omega} \subseteq Z$, then $o.t.(Z)$ is an L-cardinal$\}$. Since $\gamma \in X$ and $X \prec H_\kappa$, we have $F \in X$ and in X, F has the property as in (4). So $X \cap \gamma$ is closed under F. By the property of $F, \bar{\gamma} = o.t.(X \cap \gamma)$ is an L-cardinal.

$(3) \Rightarrow (4)$ Let $\kappa > \gamma$ be a regular cardinal such that $\{X \subseteq H_\kappa \mid$ if $X \prec H_\kappa, |X| = \omega$ and $\gamma \in X$, then $\bar{\gamma}$ is an L-cardinal$\}$ contains a club subset of $[H_\kappa]^\omega$. Let $Z = \{X \subseteq H_\kappa \mid X \prec H_\kappa, |X| = \omega$, $\gamma \in X$ and $\bar{\gamma}$ is an L-cardinal$\}$. Since $\{X \subseteq H_\kappa \mid X \prec H_\kappa, |X| = \omega$ and $\gamma \in X\}$ contains a club subset of $[H_\kappa]^\omega, Z$ contains a club subset of $[H_\kappa]^\omega$. Let $Z \supseteq D$ where D is a club subset of $[H_\kappa]^\omega$. By Fact 1.1.14(4), $D \restriction \gamma = \{X \cap \gamma \mid X \in D\}$ contains a club in $[\gamma]^\omega$. Let $D \restriction \gamma \supseteq E$ where E is a club in $[\gamma]^\omega$. By Fact 1.1.14(2), there exists $F : \gamma^{<\omega} \to \gamma$ such that $C_F \subseteq E$(i.e. if $X \subseteq \gamma, |X| = \omega$ and $F``X^{<\omega} \subseteq X$, then $X \in E$). Now suppose $X \subseteq \gamma, |X| = \omega$ and $F``X^{<\omega} \subseteq X$. We want to show that $o.t.(X)$ is an L-cardinal. Note that $X \in E$. So $X = Y \cap \gamma$ for some $Y \in D$. Note that $Y \prec H_\kappa, |Y| = \omega$, $\gamma \in Y$ and $\bar{\gamma}$ is an L-cardinal where $\bar{\gamma}$ is the image of γ under the transitive collapse of Y. Note that $\bar{\gamma} = o.t.(Y \cap \gamma)$. So $o.t.(X) = o.t.(Y \cap \gamma) = \bar{\gamma}$ is an L-cardinal. $\square$

Suppose $\gamma \geq \omega_1$ is an L-cardinal. Let $(1)'$ denote the statement "for some uncountable regular cardinal $\kappa > \gamma, \forall X((X \prec H_\kappa \wedge |X| = \omega \wedge \gamma \in X) \rightarrow \bar{\gamma}$ is not an L-cardinal)" where $\bar{\gamma}$ is the image of γ under the transitive collapse of X. Let $(2)', (3)', (4)'$ and $(5)'$ respectively be the statements which replace " is an L-cardinal" with " is not an L-cardinal" in statements $(2), (3), (4)$ and (5) in Proposition 2.3.3. The following corollary is an observation from the proof of Proposition 2.3.3.

Corollary 2.3.4 $(1)' \Leftrightarrow (2)' \Leftrightarrow (3)' \Leftrightarrow (4)' \Leftrightarrow (5)'$.

Proposition 2.3.5 *Suppose $\gamma \geq \omega_1$ is an L-cardinal and $|\gamma| = \omega_1$. Then the following are equivalent:*

(a) γ has strong reflecting property.

(b) If $\pi : \omega_1 \rightarrow \gamma$ is a bijection, then there exists a club $D \subseteq \omega_1$ such that for any $\theta \in D$, $o.t.(\{\pi(\alpha) \mid \alpha < \theta\})$ is an L-cardinal.

Proof $(a) \Rightarrow (b)$ Let κ be an uncountable regular cardinal $> \gamma$ that witnesses the strong reflecting property of γ. By Fact 2.3.2 and Fact 1.1.14(4), $\{X \cap \omega_1 \mid$

$X \prec H_\kappa \wedge |X| = \omega \wedge \gamma \in X\}$ contains a club D on ω_1. Let $\pi : \omega_1 \to \gamma$ be a bijection and $\beta \in D$. So $\beta = X \cap \omega_1$ for some X with $X \prec H_\kappa, |X| = \omega$ and $\gamma \in X$. Note that $\bar{\gamma} = o.t.(\{\pi(\alpha) \mid \alpha < X \cap \omega_1\})$ where $\bar{\gamma}$ is the image of γ under the transitive collapse of X. So $o.t.(\{\pi(\alpha) \mid \alpha < \beta\}) = \bar{\gamma}$ is an L-cardinal.

$(b) \Rightarrow (a)$ Let $\kappa > \gamma$ be a regular cardinal with $\kappa \geq (2^{\omega_1})^+$. Suppose $X \prec H_\kappa, |X| = \omega$ and $\gamma \in X$. We show that $\bar{\gamma}$ is an L-cardinal where $\bar{\gamma}$ is the image of γ under the transitive collapse of X. Since $|\gamma| = \omega_1$, let $\pi : \omega_1 \to \gamma$ be a bijection. Let $D \subseteq \omega_1$ be a witness club for π such that for any $\theta \in D$, $o.t.(\{\pi(\alpha) \mid \alpha < \theta\})$ is an L-cardinal. Note that π, D are first order definable in H_κ. So $\pi, D \in X$. Since D is unbounded in $X \cap \omega_1, X \cap \omega_1 \in D$. Since $\bar{\gamma} = o.t.(\{\pi(\alpha) \mid \alpha \in X \cap \omega_1\}), \bar{\gamma}$ is an L-cardinal. □

The following corollary is an observation from the proof of Proposition 2.3.5.

Corollary 2.3.6 *Suppose $\gamma \geq \omega_1$ is an L-cardinal and $|\gamma| = \omega_1$. Then $(1)'$ iff if $\pi : \omega_1 \to \gamma$ is a bijection, then there exists a club $D \subseteq \omega_1$ such that for any $\theta \in D$, $o.t.(\{\pi(\alpha) \mid \alpha < \theta\})$ is not an L-cardinal.*[10]

As a corollary of (1) $\Leftrightarrow$ (4) in Proposition 2.3.3, if γ has strong reflecting property, $V \subseteq N$ and $\omega_1^N = \omega_1^V$, then γ has strong reflecting property in

[10] $(1)'$ is the statement defined before Corollary 2.3.4.

N. The following proposition shows that the restriction $\omega_1^N = \omega_1^V$ is not necessary.

Proposition 2.3.7 *Suppose $\gamma \geq \omega_1$ is an L-cardinal, γ has strong reflecting property and $V \subseteq N$. Then γ has strong reflecting property in N.*

Proof Since γ has strong reflecting property in V, by Proposition 2.3.3, in V there exists $F : \gamma^{<\omega} \to \gamma$ such that for any $X \subseteq \gamma$, if X is countable and $F``X^{<\omega} \subseteq X$, then $o.t.(X)$ is an L-cardinal. Note that $F \in N$ since $V \subseteq N$. We show that in N:

For any $X \subseteq \gamma$, if X is countable and $F``X^{<\omega} \subseteq X$, then $o.t.(X)$ is an L-cardinal.

Suppose not. Then in N, there exists $\bar{\gamma} < \omega_1$ such that

(i) $\bar{\gamma}$ is not an L-cardinal and

(ii) there exists an order preserving $\pi : \bar{\gamma} \to \gamma$ such that $ran(\pi)$ is closed under F.

So, in N there exists $e : \omega \to L_{\omega_1^N}$ and $\bar{\bar{\gamma}} \in e``\omega$ such that

(a) $e``\omega \prec L_{\omega_1^N}$;

(b) $L_{\omega_1^N} \models \bar{\bar{\gamma}}$ is not an L-cardinal and

(c) there exists an order preserving $\pi : \bar{\gamma} \to \gamma$ such that $ran(\pi)$ is closed under F where $\bar{\gamma} = o.t.(e``\omega \cap \bar{\bar{\gamma}})$.

Let $\langle \varphi_i \mid i \in \omega \rangle$ be a recursive enumeration of formulas with infinite repetitions. We assume that for any $i \in \omega, \varphi_i$ has free variables among $x_0, \cdots, x_{i+1}$. So in N there exist $e : \omega \to L_{\omega_1^N}, \pi : \omega \to \gamma$ and $\bar{\bar{\gamma}} \in e``\omega$ such that

(1) for any $i \in \omega$, if there exists $a \in L_{\omega_1^N}$ such that $L_{\omega_1^N} \models \varphi_i[a, e(0), \cdots, e(i)]$, then $L_{\omega_1^N} \models \varphi_i[e(2i+1), e(0), \cdots, e(i)]$;

(2) $ran(\pi)$ is closed under F;

(3) $L_{\omega_1^N} \models \bar{\bar{\gamma}}$ is not an L-cardinal and

(4) for any $i \in \omega$, if $e(i) \notin \bar{\bar{\gamma}}$, then $\pi(i) = 0$ and for any $i < j \in \omega$, if $e(i), e(j) \in \bar{\bar{\gamma}}$, then $\pi(i) < \pi(j) \Leftrightarrow e(i) < e(j)$ and $\pi(i) = \pi(j) \Leftrightarrow e(i) = e(j)$.

In N, let $T = \{(e \restriction n, \pi \restriction n) : e$ and π have properties $(1) - (4)\}$. By the definition of T, T is a tree. Note that from $(1) - (4), T \in V$ by absoluteness. Since in N, there exists (e, π) with properties $(1) - (4)$, T has an infinite branch in N. By absoluteness, T has an infinite branch in V and such a branch corresponds to the existence of (e, π) with properties $(1) - (4)$ in V. So in V, there exists $X \subseteq \gamma$ such that X is countable, $F``X^{<\omega} \subseteq X$ and $o.t.(X)$ is not an L-cardinal. Contradiction. $\square$

Proposition 2.3.8 *Suppose $\gamma \geq \omega_1$ is an L-cardinal. Then the following are equivalent:*

(a) γ has weakly reflecting property.

(b) For any uncountable regular cardinal $\kappa > \gamma$,

$$\exists X(X \prec H_\kappa \wedge |X| = \omega \wedge \gamma \in X \wedge \bar{\gamma} \text{ is an } L\text{-cardinal})$$

where $\bar{\gamma}$ is the image of γ under the transitive collapse of X.

(c) For some uncountable regular cardinal $\kappa > \gamma, \{X \subseteq H_\kappa \mid$ if $X \prec H_\kappa, |X| = \omega$ and $\gamma \in X$, then $\bar{\gamma}$ is an L-cardinal$\}$ is a stationary subset of $[H_\kappa]^\omega$ where $\bar{\gamma}$ is the image of γ under the transitive collapse of X.

(d) For any $F : \gamma^{<\omega} \to \gamma$, there exists $X \subseteq \gamma$ such that X is countable, $F``X^{<\omega} \subseteq X$ and $o.t.(X)$ is an L-cardinal.

(e) For any uncountable regular cardinal $\kappa > \gamma, \{X \subseteq H_\kappa \mid$ if $X \prec H_\kappa, |X| = \omega$ and $\gamma \in X$, then $\bar{\gamma}$ is an L-cardinal$\}$ is a stationary subset of $[H_\kappa]^\omega$ where $\bar{\gamma}$ is the image of γ under the transitive collapse of X.

Proof Note that $(e) \Rightarrow (c)$ and $(c) \Rightarrow (a)$. It suffices to show that $(a) \Rightarrow (d), (d) \Rightarrow (b)$ and $(b) \Rightarrow (e)$.

$(a) \Rightarrow (d)$: Assume (a) holds. Suppose there exists $F : \gamma^{<\omega} \to \gamma$ such that for any $X \subseteq \gamma$, if X is countable and $F``X^{<\omega} \subseteq X$, then $o.t.(X)$ is not an

L-cardinal. By $(4)' \Leftrightarrow (2)'$ in Corollary 2.3.4, for any uncountable regular cardinal $\kappa > \gamma$,

$$\forall X((X \prec H_\kappa \wedge |X| = \omega \wedge \gamma \in X) \rightarrow \bar{\gamma} \text{ is not an } L\text{-cardinal}).$$

Contradiction.

$(d) \Rightarrow (b)$: Assume (d) holds. Suppose for some uncountable regular cardinal $\kappa > \gamma$,

$$\forall X((X \prec H_\kappa \wedge |X| = \omega \wedge \gamma \in X) \rightarrow \bar{\gamma} \text{ is not an } L\text{-cardinal}).$$

By By $(1)' \Leftrightarrow (4)'$ in Corollary 2.3.4, there exists $F : \gamma^{<\omega} \rightarrow \gamma$ such that for any $X \subseteq \gamma$, if X is countable and $F``X^{<\omega} \subseteq X$, then $o.t.(X)$ is not an L-cardinal. Contradiction.

$(b) \Rightarrow (e)$: Suppose (b) holds and (e) does not hold. Then for some uncountable regular cardinal $\kappa > \gamma$, $\{X \subseteq H_\kappa \mid$ if $X \prec H_\kappa, |X| = \omega$ and $\gamma \in X$, then $\bar{\gamma}$ is an L-cardinal$\} \cap C = \emptyset$ for some C which is a club subset of $[H_\kappa]^\omega$. So $C \subseteq \{X \subseteq H_\kappa \mid X \prec H_\kappa, |X| = \omega$, $\gamma \in X$ and $\bar{\gamma}$ is not an L-cardinal$\}$. By $(3)' \Leftrightarrow (1)'$ in Corollary 2.3.4, we have (b) does not hold. $\square$

Suppose $\gamma \geq \omega_1$ is an L-cardinal. As a corollary of $(a) \Leftrightarrow (d)$ in Proposition 2.3.8, if γ has weakly reflecting property, $N \subseteq V$ and $\omega_1^N = \omega_1^V$, then γ has weakly reflecting property in N. By the similar argument as in Proposition 2.3.7, if γ has weakly reflecting property and $N \subseteq V$, then γ has weakly reflecting property in N.

Proposition 2.3.9 *Suppose $\gamma \geq \omega_1$ is an L-cardinal and $|\gamma| = \omega_1$. Then the following are equivalent:*

(1) γ has weakly reflecting property.

(2) For some bijection $\pi : \omega_1 \to \gamma$, there exists a stationary $D \subseteq \omega_1$ such that for any $\theta \in D$, $o.t.(\{\pi(\alpha) \mid \alpha < \theta\})$ is an L-cardinal.

Proof (1) $\Rightarrow$ (2) Let $\kappa \geq (2^{\omega_1})^+$ be an uncountable regular cardinal $> \gamma$ that witnesses the weakly reflecting property of γ. Suppose (2) does not hold. Then for any bijection $\pi : \omega_1 \to \gamma, S = \{\delta < \omega_1 : o.t.(\{\pi(\alpha) \mid \alpha < \delta\})$ is an L-cardinal$\}$ is not stationary. So for any bijection $\pi : \omega_1 \to \gamma$ there exists a club $D \subseteq \omega_1$ such that for any $\theta \in D, o.t.(\{\pi(\alpha) \mid \alpha < \theta\})$ is not an L-cardinal. By Corollary 2.3.6 and $(1)' \Leftrightarrow (2)'$ in Corollary 2.3.4, $\forall X((X \prec H_\kappa \wedge |X| = \omega \wedge \gamma \in X) \to \bar{\gamma}$ is not an L-cardinal). But we have $\exists X(X \prec H_\kappa \wedge |X| = \omega \wedge \gamma \in X \wedge \bar{\gamma}$ is an L-cardinal). Contradiction.

(2) $\Rightarrow$ (1): Let $\kappa > \gamma$ be a regular cardinal. Let $\pi : \omega_1 \to \gamma$ be a witness bijection and S be a witness stationary set for π such that for any $\theta \in S$, $o.t.(\{\pi(\alpha) \mid \alpha < \theta\})$ is an L-cardinal. Suppose $\forall X((X \prec H_\kappa \wedge |X| = \omega \wedge \gamma \in X) \to \bar{\gamma}$ is not an L-cardinal). By Corollary 2.3.6, for π there exists a club D on ω_1 such that for any $\theta \in D$, $o.t.(\{\pi(\alpha) \mid \alpha < \theta\})$ is not an L-cardinal. Since S is stationary, $S \cap D \neq \emptyset$. Contradiction. $\square$

Definition 2.3.10 *Suppose $\gamma \geq \omega_1$ is an L-cardinal and $|\gamma| = \omega_1$. γ has* *reflecting property if and only if for some bijection $\pi : \omega_1 \to \gamma$, there exists a club $D \subseteq \omega_1$ such that for any $\theta \in D$, $o.t.(\{\pi(\alpha) \mid \alpha < \theta\})$ is an L-cardinal.*

Proposition 2.3.11 *Suppose $\gamma \geq \omega_1$ is an L-cardinal and $|\gamma| = \omega_1$. Then the following are equivalent:*

(1) γ has reflecting property.

(2) γ has strong reflecting property.

Proof Note that (2) $\Rightarrow$ (1) by Proposition 2.3.5. It suffices to show that (1) $\Rightarrow$ (2). Let $\kappa > \gamma$ be a regular cardinal with $\kappa \geq (2^{\omega_1})^+$. Suppose γ has reflecting property, $X \prec H_\kappa, |X| = \omega$ and $\gamma \in X$. Let $\bar{\gamma}$ be the image of γ under the transitive collapse of X. We show that $\bar{\gamma}$ is an L-cardinal. Since γ has reflecting property, there exist a bijection $f : \omega_1 \to \gamma$ and a club D on ω_1 such that for any $\theta \in D$, $o.t.(\{f(\alpha) \mid \alpha < \theta\})$ is an L-cardinal. Since the properties of f, D are first order definable in H_κ, we can take $f, D \in X$. Since D is unbounded in $X \cap \omega_1$, $X \cap \omega_1 \in D$. Note that $\bar{\gamma} = o.t.(\{f(\alpha) \mid \alpha < X \cap \omega_1\})$. So $\bar{\gamma}$ is an L-cardinal. $\square$

Proposition 2.3.12 *If $\omega_1 \leq \gamma_0 < \gamma_1$ are L-cardinals and γ_1 has strong reflecting property (weakly reflecting property), then γ_0 has strong reflecting property (weakly reflecting property).*

Proof It suffices to show the case for strong reflecting property. Let $\kappa > \gamma_1$ be a regular cardinal. It suffices to show if $X \prec H_\kappa, |X| = \omega$ and $\{\gamma_0, \gamma_1\} \subseteq X$, then $\bar{\gamma_0}$ is an L-cardinal where $\bar{\gamma_0}, \bar{\gamma_1}$ are images of γ_0, γ_1 under the transitive collapse of X. Since γ_0 is an L-cardinal and $\gamma_0 < \gamma_1, L_{\gamma_1} \models \gamma_0$ is a cardinal. Since $\gamma_1 \in X, L_{\gamma_1} \in X$. Since $\bar{L_{\gamma_1}} = L_{\bar{\gamma_1}}$ and $\bar{L_{\gamma_1}} \models \bar{\gamma_0}$ is a cardinal, $L_{\bar{\gamma_1}} \models \bar{\gamma_0}$ is a cardinal. Since γ_1 has strong reflecting property, $\bar{\gamma_1}$ is an L-cardinal. So $\bar{\gamma_0}$ is an L-cardinal. $\square$

Proposition 2.3.13 *Suppose $\gamma \geq \omega_1$ is an L-cardinal and has strong reflecting property. $\kappa > \gamma$ is a regular cardinal. Suppose $Z \prec H_\kappa$, $|Z| \leq \omega_1$ and $\gamma \in Z$. Then $\bar{\gamma}$ is an L-cardinal where $\bar{\gamma}$ is the images of γ under the transitive collapse of Z.*

Proof Let M be the transitive collapse of Z and $\pi : M \prec H_\kappa$ be the collapsing map. By the definition of $\bar{\gamma}$, $\pi(\bar{\gamma}) = \gamma$. We show that $\bar{\gamma}$ is an L-cardinal. Suppose $\bar{\gamma}$ is not an L-cardinal and we try to get a contradiction. Since $|M| \leq \omega_1$ and M is transitive, $M \in H_\kappa$. Take $Y \prec H_\kappa$ such that $|Y| = \omega$ and $M, \bar{\gamma} \in Y$. Since $\bar{\gamma} \in M$, we have $\bar{\gamma} \subseteq M$ and $|\bar{\gamma}| \leq \omega_1$. Since $H_\kappa \models$ "$\bar{\gamma}$ is not an L-cardinal", $Y \models$ "$\bar{\gamma}$ is not an L-cardinal". Let N be the transitive collapse of Y and $\bar{\bar{\gamma}}$ be the image of $\bar{\gamma}$ under the transitive collapse of Y. Then $N \models$ "$\bar{\bar{\gamma}}$ is not an L-cardinal". So, $\bar{\bar{\gamma}}$ is not an L-cardinal. Let $X = \pi``(Y \cap M)$. Since $\bar{\gamma} \in Y \cap M$ and $\pi(\bar{\gamma}) = \gamma$, we have $\gamma \in X$. Since

$X \prec H_\kappa, X \subseteq Z$ and $Z \prec H_\kappa$, we have $X \prec Z \prec H_\kappa$. Since $\bar{\gamma} \in Y \cap M$ and $\bar{\bar{\gamma}}$ is the image of $\bar{\gamma}$ under the transitive collapse of Y, the image of γ under the transitive collapse of X is $\bar{\bar{\gamma}}$. Since X is countable and γ has strong reflecting property, $\bar{\bar{\gamma}}$ is an L-cardinal. Contradiction. □

Proposition 2.3.14 *The following are equivalent:*

(1) ω_2 has strong reflecting property.

(2) ω_2 is a limit cardinal in L and for any L-cardinal $\omega_1 \leq \gamma < \omega_2, \gamma$ has strong reflecting property.

Proof (1) $\Rightarrow$ (2) We show that if ω_2 has strong reflecting property, then ω_2 is a limit cardinal in L. The rest part follows from Proposition 2.3.12. Let $\kappa > \omega_2$ be the regular cardinal that witnesses the strong reflecting property of ω_2. Fix $\alpha < \omega_2$, choose $Z \prec H_\kappa$ such that $|Z| = \omega_1$, $\alpha \subseteq Z$ and $\omega_2 \in Z$. By Proposition 2.3.13, $\bar{\omega_2}$ is an L-cardinal where $\bar{\omega_2}$ is the image of ω_2 under the transitive collapse of Z. Note that $\alpha \leq \bar{\omega_2} < \omega_2$. So ω_2 is a limit cardinal in L.

(2) $\Rightarrow$ (1) Suppose $X \prec H_\kappa$ for some regular cardinal $\kappa > \omega_2, |X| = \omega$ and $\omega_2 \in X$. Let $\bar{\omega_2}$ be the image of ω_2 under the transitive collapse of X. We show that $\bar{\omega_2}$ is an L-cardinal. Note that $\bar{\omega_2} = o.t.(X \cap \omega_2)$. Let $E = \{\gamma \mid \omega_1 \leq \gamma < \omega_2 \wedge \gamma \text{ is an } L\text{-cardinal}\}$. Note that E is definable in H_κ.

Since ω_2 is a limit cardinal in L, E is cofinal in ω_2 and hence $E \cap X$ is cofinal in $\omega_2 \cap X$. Note that for any $\gamma \in E \cap X, \bar{\gamma} = o.t.(X \cap \gamma)$ where $\bar{\gamma}$ is the image of γ under the transitive collapse of X. So $\bar{\omega_2} = sup(\{\bar{\gamma} \mid \gamma \in E \cap X\})$. By Proposition 2.3.13, for any $\gamma \in E \cap X, \bar{\gamma}$ is an L-cardinal. So $\bar{\omega_2}$ is an L-cardinal. $\square$

Theorem 2.3.15 *Assume γ is an L-cardinal and has strong reflecting property. Suppose $\gamma > \omega_2$. Then $0^\sharp$ exists.*

Proof Since $\gamma > \omega_2$ has strong reflecting property, ω_2 has strong reflecting property. Let $\kappa > \omega_2$ be a regular cardinal that witnesses the strong reflecting property of ω_2.

Case 1: CH holds. Take $Z \prec H_\kappa$ such that $|Z| = \omega_1$, $\{\omega_2, \gamma\} \subseteq Z$ and $Z^\omega \subseteq Z$. Let $\bar{\gamma}, \bar{\omega_2}$ be the images of γ, ω_2 under the transitive collapse of Z. By Proposistion 2.3.13, $\bar{\gamma}, \bar{\omega_2}$ are L-cardinals. Let M be the transitive collapse of Z and $j : M \prec H_\kappa$ be the collapsing map. Since $\gamma \in Z$ and $|Z| = \omega_1$, j is not the identity map. Let $\lambda = crit(j)$. Then $\lambda \leq \bar{\omega_2}$. The rest is similar to the proof of "**Harrington's** $\star$ implies $0^\sharp$ exists" in Theorem 2.0.1. Define

$$U = \{X \subseteq \lambda \mid X \in L \wedge \lambda \in j(X)\}.$$

Note that $U \subseteq L^M$ since $\bar{\gamma}$ is an L-cardinal. As in Theorem 2.0.1, U is an L-ultrafilter on λ. From U we can define the ultrapower model L^λ/U. By the similar argument as in Theorem 2.0.1, L^λ/U is well founded. So $0^\sharp$ exists.

Case 2: CH does not hold.[11] Build an elementary chain $\langle Z_\alpha \mid \alpha < \omega_1 \rangle$ of submodels of H_κ such that for all $\alpha < \beta < \omega_1$,

(1) $Z_\alpha \prec Z_\beta \prec H_\kappa$;

(2) $Z_\alpha \in Z_\beta$ and $|Z_\alpha| = \omega$;

(3) $\{\gamma, \omega_2\} \subseteq Z_0$.

Let $Z = \bigcup_{\alpha<\omega_1} Z_\alpha$. Then $|Z| = \omega_1$ and $\gamma, \omega_2 \in Z \prec H_\kappa$. Let M be the transitive collapse of Z and $\pi : M \prec H_\kappa$ be the collapsing map. Let M_α be the transitive collapse of Z_α and $\pi_\alpha : M_\alpha \prec H_\kappa$ be the collapsing map. Since $Z_\alpha \prec Z$, let $j_\alpha : M_\alpha \prec M$ be the induced elementary embedding. Then $\pi_\alpha = \pi \circ j_\alpha$. Since $Z = \bigcup_{\alpha<\omega_1} Z_\alpha, M = \bigcup_{\alpha<\omega_1} j_\alpha``M_\alpha$. Note that $\bar{\omega_2} < \bar{\gamma} \in M, \pi(\bar{\omega_2}) = \omega_2$, $\pi(\bar{\gamma}) = \gamma$ and by Proposistion 2.3.13, $\bar{\gamma}, \bar{\omega_2}$ are L-cardinals.[12] Since $\omega_1 \subseteq Z$, $\bar{\omega_1} = \omega_1^M = \omega_1$ and $crit(\pi) > \bar{\omega_1}$. Since $\omega_2 \in Z$, $crit(\pi) \leq \bar{\omega_2}$. So $crit(\pi) = \bar{\omega_2}$.

[11] Since CH does not hold, it is possible that $2^\omega \geq \gamma$. Suppose $Z \prec H_\kappa, \gamma \in Z$ and $Z^\omega \subseteq Z$ as in Case 1. Note that $\mathbb{R} \subseteq Z$. Since there exists surjective $f : \mathbb{R} \to \gamma$, we can take such f in Z. So $\gamma \subseteq Z$. Suppose M is the transitive collapse of Z. Then $\gamma \subseteq M$. This example explains that if CH does not hold, the method in Case 1 does not work.

[12] $\bar{\gamma}, \bar{\omega_2}$ are the images of γ, ω_2 under the transitive collapse of Z.

Since $\bar{\omega_2}$ is an L-cardinal and $\bar{\omega_2} < \bar{\gamma}, \mathcal{P}(\bar{\omega_2}) \cap L \subseteq L_{\bar{\gamma}} \subseteq M$ and $\mathcal{P}(\bar{\omega_2}) \cap L \in M$. Define

$$U = \{X \subseteq \bar{\omega_2} \mid X \in L \wedge \bar{\omega_2} \in \pi(X)\}.$$

U is an L-ultrafilter and $U \subseteq L^M$. For any $\alpha < \omega_1, Z_\alpha \in Z$, $Z \models$ "Z_α is countable" and the image of Z_α under the transitive collapse of Z is j_α"M_α. So for any $\alpha < \omega_1, j_\alpha$"$M_\alpha \in M, M \models$ "j_α"M_α is countable" and $j_\alpha \in M$. Note that ω_2 is a limit cardinal in L since ω_2 has strong reflecting property.

We assume that $0^\sharp$ does not exist and try to get a contradiction.

Lemma 2.3.16 *U is ω_1-complete.*

Proof We show that if $Y \subseteq U$ and Y is countable, then $\bigcap Y \neq \emptyset$. Since $Y \subseteq M$, take $\alpha < \omega_1$ large enough such that $Y \subseteq j_\alpha$"M_α. Let

$$S = \mathcal{P}(\bar{\omega_2}) \cap L \cap j_\alpha\text{“}M_\alpha.$$

$S \in M$ since $\{j_\alpha$"$M_\alpha, \mathcal{P}(\bar{\omega_2}) \cap L\} \subseteq M$. So $M \models$ "$S \subseteq \mathcal{P}(\bar{\omega_2}) \cap L$ and S is countable". Consider the following covering property $\triangle$:

For any $S \subseteq Ord$ with $|S| \leq \omega_1$, there exists $T \supseteq S$ such that $T \in L$ and $|T| \leq \omega_1$.

We know that if $0^\sharp$ does not exist, then the covering property $\triangle$ holds. Since property $\triangle$ is first order definable in $H_\kappa, H_\kappa \models \triangle$. So $M \models \triangle$. Hence $M \models \exists T(T \subseteq \mathcal{P}(\bar{\omega_2}) \cap L \wedge T \supseteq S \wedge T \in L \wedge |T| \leq \omega_1)$. Fix $T \in M$ such that $T \subseteq \mathcal{P}(\bar{\omega_2}) \cap L, T \supseteq S, T \in L$ and $|T| \leq \omega_1$.

Note that $\omega_1 \subseteq Z$ and $\omega_1^M = \omega_1$. Since $\bar{\omega_2} = crit(\pi) > \omega_1, \pi(T) = \pi``T$. Since ω_2 is a limit cardinal in L, $M \models \bar{\omega_2}$ is a limit L-cardinal.[13] Since $T \in M, \mathcal{P}(T) \cap L \in M$. Since $M \models$ "$T \in L$ and $|T| \leq \omega_1$" and $\bar{\omega_2} = \omega_2^M$, we have $M \models |T|^L < \bar{\omega_2}$. Since $M \models \bar{\omega_2}$ is a limit L-cardinal, $M \models$ "$|\mathcal{P}(T) \cap L| = \omega_1$". So $\pi(\mathcal{P}(T) \cap L) = \pi``(\mathcal{P}(T) \cap L)$. Note that U is not a definable subset of M.

Claim $U \cap T \in M$.

Proof Since $\pi(T) = \pi``T, \pi``(U \cap T) = \{\pi(A) \mid A \in T \wedge \bar{\omega_2} \in \pi(A)\} = \{B \in \pi(T) \mid \bar{\omega_2} \in B\}$. Since $\pi(T) \in L, \pi``(U \cap T) \in L$. So $\pi``(U \cap T) \in \mathcal{P}(\pi``T) \cap L$. Note that $\mathcal{P}(\pi``T) \cap L = \pi``(\mathcal{P}(T) \cap L)$ since for all $D \in \mathcal{P}(T) \cap L, \pi(D) = \pi``D$. So $\pi``(U \cap T) \in \pi``(\mathcal{P}(T) \cap L)$ and $\pi``(U \cap T) = \pi(D) = \pi``D$ for some $D \in \mathcal{P}(T) \cap L$. Hence $U \cap T = D \in \mathcal{P}(T) \cap L \subseteq M$ and $U \cap T \in M$. $\square$

Since $U \cap T \in M, \pi(U \cap T) = \pi``(U \cap T)$. Note that $Y \subseteq j_\alpha``M_\alpha \cap \mathcal{P}(\bar{\omega_2}) \cap L = S \subseteq T$. Since $Y \subseteq T \cap U$, to show that $\bigcap Y \neq \emptyset$, it suffices to show that

[13] In fact, since $\bar{\omega_2}$ is an L-cardinal, $\bar{\omega_2}$ is a limit L-cardinal.

$\bigcap(U \cap T) \neq \emptyset$. Note that $\bar{\omega_2} \in \bigcap \pi``(U \cap T)$. Since $\bigcap \pi``(U \cap T) \neq \emptyset$ and $\pi(U \cap T) = \pi``(U \cap T)$, we have $\bigcap \pi``(U \cap T) = \bigcap \pi(U \cap T) = \pi(\bigcap(U \cap T)) \neq \emptyset$. So $\bigcap(U \cap T) \neq \emptyset$. $\square$

From U we can define the ultrapower model $L^{\bar{\omega_2}}/U$. Since U is ω_1-complete, $L^{\bar{\omega_2}}/U$ is well founded. So $0^\sharp$ exists which contradicts our assumption that $0^\sharp$ does not exist. $\square$

Corollary 2.3.17 *The following are equivalent:*

(i) There exists $\gamma > \omega_2$ such that γ is an L-cardinal and γ has strong reflecting property.

(ii) $0^\sharp$ exists.

(iii) ω_3 has strong reflecting property.

(iv) For any $\gamma \geq \omega_1$, if γ is an L-cardinal, then γ has strong reflecting property.

Proof It suffices to show that if $0^\sharp$ exists and $\gamma \geq \omega_1$ is an L-cardinal, then γ has strong reflecting property. Suppose $\kappa > \gamma$ is a regular cardinal, $X \prec H_\kappa, |X| = \omega$, $\gamma \in X$ and $\gamma \geq \omega_1$ is an L-cardinal. We show that $\bar{\gamma}$ is an L-cardinal where $\bar{\gamma}$ is the image of γ under the transitive collapse of X. Since $\gamma \in X$ and $0^\sharp \in X, \mathcal{M}(0^\sharp, \gamma + 1) \in X$.[14] Note that for any

[14]Note that $\mathcal{M}(0^\sharp, \alpha)$ is the unique transitive $(0^\sharp, \alpha)$ model.

$\alpha \in Ord, \mathcal{M}(0^\sharp, \alpha) \prec L$. Since $L \models$ "γ is a cardinal" and $\gamma \in \mathcal{M}(0^\sharp, \gamma + 1), \mathcal{M}(0^\sharp, \gamma+1) \models \gamma$ is a cardinal. So $\mathcal{M}(0^\sharp, \gamma+1)^* \models$ "$\bar{\gamma}$ is a cardinal" where $\mathcal{M}(0^\sharp, \gamma+1)^*$ is the image of $\mathcal{M}(0^\sharp, \gamma+1)$ under the transitive collapse of X. Note that $\mathcal{M}(0^\sharp, \gamma+1)^* = \mathcal{M}(0^\sharp, \bar{\gamma}+1)$. Since $\mathcal{M}(0^\sharp, \bar{\gamma}+1) \models$ "$\bar{\gamma}$ is a cardinal" and $\mathcal{M}(0^\sharp, \bar{\gamma}+1) \prec L, L \models \bar{\gamma}$ is a cardinal. □

2.3.2 Baumgartner's forcing

In this section we introduce Baumgartner's forcing P_S^B and prove some properties of P_S^B which will be used in Section 2.3.7. In this section, we assume that S is a stationary subset of ω_1.

Definition 2.3.18 *Define* $P_S^B = \{f : dom(f) \to S \mid dom(f) \subseteq \omega_1$ *is finite and* $\exists \alpha > \max(dom(f))\, \exists g : \alpha \to S($*$g$ is continuous, increasing and* $g \restriction dom(f) = f$*)* $\}$.

For $f, g \in P_S^B, g \leq f$ if and only if $f \subseteq g$. Note that the following are equivalent:

(1) $f \in P_S^B$.

(2) $dom(f) \subseteq \omega_1$ is finite and there exists $g : \alpha + 1 \to S$ such that g is continuous, increasing and $g \restriction dom(f) = f$ where $\alpha = \max(dom(f))$.

(3) $dom(f) \subseteq \omega_1$ is finite and there exists $C \subseteq S$ such that C is closed, $o.t.(C) = \alpha + 1$ and for any $\beta \in dom(f), f(\beta)$ is the β-th element of C

where $\alpha = \max(dom(f))$.

Let G be P_S^B-generic over V. Define $F_G = \bigcup\{f \mid f \in G\}$. Since $dom(f) \subseteq \omega_1$ is finite for any $f \in P_S^B$, from the definition of P_S^B, it is not difficult to check that:

- For any $f \in P_S^B$ and all $\alpha < \omega_1$, there exists $g \in P_S^B$ such that $g \leq f$ and $\alpha \in dom(g)$.
- For any $f \in P_S^B$, for any $\alpha \in dom(f)$, if α is a limit ordinal, then for any $\eta < f(\alpha)$, there exist $g \in P_S^B$ and $\beta < \alpha$ such that $g \leq f, \beta \in dom(g)$ and $g(\beta) > \eta$.

Note that if $F : \omega_1 \to S$ is increasing and continuous, then $ran(F) \subseteq S$ is a club on ω_1. If $C \subseteq S$ is a club on ω_1, then $F : \omega_1 \to S$ is increasing and continuous where $F(\alpha) =$ the α-th element of C.

So $F_G : \omega_1 \to S$ is increasing and continuous. Let $C = ran(F_G)$. Then $C \subseteq S$ is a club on ω_1. Let $D = \{\alpha \mid \alpha$ is a limit point of $C\}$. Note that $D \subseteq C$ is a club on ω_1.

For $f \in P_S^B$, define $(P_S^B)_f = \{g \in P_S^B \mid g \leq f$ and $\max(dom(g)) = \max(dom(f))\}$. Note that $|(P_S^B)_f| = \omega$ for $f \in P_S^B$.

Lemma 2.3.19 (Z_3) *If $f \in P_S^B$, then $f \Vdash$ "$\dot{G} \cap (P_S^B)_f$ is $(P_S^B)_f$-generic over V". Equivalently, if G is P_S^B-generic over V and $f \in G$, then $G \cap (P_S^B)_f$ is $(P_S^B)_f$-generic over V.*

Proof We first show that for any $h \leq f$, if D is a dense subset of $(P_S^B)_f$, then there exists $p \in D$ such that $h \cup p \in P_S^B$. Suppose $h \leq f$ and $D \subseteq (P_S^B)_f$ is dense. Let $\max(dom(f)) = \beta$. Since $h \leq f, h \restriction (\beta+1) \in (P_S^B)_f$. Since $D \subseteq (P_S^B)_f$ is dense, take $p \in D$ such that $p \leq h \restriction (\beta+1)$.

Claim

$$h \cup p \in P_S^B.$$

Proof Let $\alpha = \max(dom(h))$. Since $h \in P_S^B$, there exists $E \subseteq S$ such that E is closed, $o.t.(E) = \alpha+1$ and for any $\gamma \in dom(h), h(\gamma)$ is the γ-th element of E. Since $h \leq f$, $h(\beta) = f(\beta)$. Since $p \in (P_S^B)_f \subseteq P_S^B, \max(dom(p)) = \beta$. Let $F \subseteq S$ be closed such that $o.t.(F) = \beta+1$ and for any $\gamma \in dom(p), p(\gamma)$ is the γ-th element of F. Note that $p(\beta) = f(\beta)$. Let $C = \{\gamma \in E \mid \gamma \geq h(\beta) = f(\beta) = p(\beta)\} \cup F$. Since E, F are closed, $C \subseteq S$ is closed. Since $o.t.(E) = \alpha+1$, $o.t.(F) = \beta+1$ and $p \leq h \restriction (\beta+1)$, we have $o.t.(C) = \alpha+1$. From the property of h, p and the definition of C, for any $\gamma \in dom(h \cup p), (h \cup p)(\gamma)$ is the γ-th element of C. So $h \cup p \in P_S^B$. $\square$

Suppose G is P_S^B-generic over V and $f \in G$. We show that $G \cap (P_S^B)_f$ is $(P_S^B)_f$-generic over V. Suppose not. Then there exists $D \in V$ such that D is a dense subset of $(P_S^B)_f$ and $G \cap D = \emptyset$. So there exists $h \in G$ such that $h \leq f$ and $h \Vdash \dot{G} \cap D = \emptyset$. We have shown that for such h, there exists $p \in D$ such that $h \cup p \in P_S^B$. So $h \cup p \Vdash p \in \dot{G} \cap D$. Contradiction. $\square$

Fact 2.3.20 *(Folklore, [9]) (Z_3) Suppose $S \subseteq \omega_1$ is stationary. Then for any $\alpha < \omega_1$, there exists a closed set $C \subseteq S$ such that $o.t.(C) = \alpha$.*

Lemma 2.3.21 *For any $\alpha < \omega_1$ and any $\beta < \omega_1$, there exists $C \subseteq S$ such that C is closed, $\alpha < o.t.(C) < \omega_1$ and $\min(C) > \beta$.*

Proof Prove by induction on $\alpha < \omega_1$. Suppose $\alpha = \gamma + 1$ and $\beta < \omega_1$. Choose $C \subseteq S$ such that C is closed, $\gamma < o.t.(C) < \omega_1$ and $\min(C) > \beta$. Since S is unbounded in ω_1, take $\eta \in S$ with $\eta > \sup(C)$. Let $D = C \cup \{\eta\}$. Then $D \subseteq S$ is closed, $\alpha < o.t.(D) < \omega_1$ and $\min(D) > \beta$.

Suppose α is a limit ordinal and we assume that the conclusion holds for all $\alpha' < \alpha$. Take an increasing sequence $\langle \alpha_i \mid i \in \omega\rangle$ which is cofinal in α. Fix $\beta < \omega_1$. Take $X \prec H_{\omega_2}$ such that $|X| = \omega, \{\alpha, \beta, S\} \subseteq X$ and $X \cap \omega_1 \in S$. Such X exists since S is stationary. Fix an increasing sequence $\langle \beta_i \mid i \in \omega\rangle$ such that $\beta_0 > \beta$ and $X \cap \omega_1 = \sup(\{\beta_i \mid i \in \omega\})$. By induction hypothesis, choose $C_0 \in X$ such that $C_0 \subseteq S$ is closed, $\alpha_0 < o.t.(C_0) < \omega_1$ and $\min(C_0) > \beta_0$. Suppose $C_i \in X$ is given. Choose $C_{i+1} \in X$ such that $C_{i+1} \subseteq S$ is closed, $\alpha_{i+1} < o.t.(C_{i+1}) < \omega_1, \min(C_{i+1}) > \beta_{i+1}$ and $\min(C_{i+1}) > \sup(C_i)$. Let $C = \bigcup_{i\in\omega} C_i \cup \{X \cap \omega_1\}$. Note that $\sup(\bigcup_{i\in\omega} C_i) = X \cap \omega_1, \sup(C) = X \cap \omega_1, C \subseteq S, C$ is closed, $\alpha < o.t.(C) < \omega_1$ and $\min(C) > \beta$. □

Lemma 2.3.22 *Suppose $X \prec H_{\omega_2}, |X| = \omega, S \in X$ and $X \cap \omega_1 \in S$. Let $\delta = X \cap \omega_1$. Then there exists $C \subseteq S$ such that C is closed, $C \subseteq \delta + 1$ and*

$o.t.(C) = \delta + 1$. So δ is the δ-th element of C.

Proof By Lemma 2.3.21, for all $\beta < \delta$ and all $\eta < \delta$, there exists closed $C \subseteq S$ such that $C \in X, o.t.(C) > \beta$ and $\min(C) > \eta$.

Choose an increasing sequence $\langle \delta_i \mid i \in \omega \rangle$ such that $\sup(\{\delta_i \mid i \in \omega\}) = \delta$. Choose closed $C_0 \subseteq S$ such that $C_0 \in X$ and $o.t.(C) > \delta_0$. Given C_i, choose closed $C_{i+1} \subseteq S$ such that $C_{i+1} \in X, o.t.(C_{i+1}) > \delta_{i+1}$ and $\sup(C_i) < \min(C_{i+1})$. Let $C = \bigcup_{i\in\omega} C_i \cup \{\delta\}$. Then $C \subseteq S, C \subseteq \delta + 1, \sup(\bigcup_{i\in\omega} C_i) = \delta, C$ is closed and $o.t.(C) = \delta + 1$. □

Definition 2.3.23 *A limit ordinal γ is indecomposable if and only if there exist no $\alpha < \gamma$ and $\beta < \gamma$ such that $\alpha + \beta = \gamma$ if and only if $\alpha + \gamma = \gamma$ for any $\alpha < \gamma$ if and only if $\gamma = \omega^\alpha$ for some α (this is ordinal exponentiation).*

Note that if γ is indecomposable, then for any $\alpha < \gamma, o.t.(\{\beta \mid \alpha \leq \beta < \gamma\}) = \gamma$.

For any $\eta < \omega_1$, define $P^B_S \restriction \eta = \{f \in P^B_S \mid (dom(f) \cup ran(f)) \subseteq \eta\}$.

Lemma 2.3.24 (Z_3) *Suppose $\eta < \omega_1$ is indecomposable and $\{(\eta, \eta)\} \in P^B_S$. Let $f = \{(\eta, \eta)\}$. Then*

$$(P^B_S)_f = \{g \cup \{(\eta, \eta)\} \mid g \in P^B_S \restriction \eta\}.$$

Proof $\subseteq$ is trivial. Fix $g \in P^B_S \restriction \eta$. We show that $g \cup \{(\eta, \eta)\} \in P^B_S$. It suffices to show that there exists an increasing continuous $H : \eta + 1 \to$

$S \cap (\eta+1)$ such that H extends g and $H(\eta) = \eta$. Let $\xi = \max(dom(g))$. Since $g \in P_S^B$, there exists an increasing continuous $F : \xi+1 \to S \cap (g(\xi)+1)$ such that F extends g. Let $E : \eta+1 \to S \cap (\eta+1)$ be increasing and continuous with $E(\eta) = \eta$. Such E exists since $\{(\eta, \eta)\} \in P_S^B$. Let $C = ran(E) \setminus (g(\xi) + 1)$. Note that $C \subseteq S$ is closed and $o.t.(C) = o.t.((\eta + 1) \setminus (g(\xi) + 1)) = \eta + 1$ since $g(\xi) < \eta$. Let $\pi : \eta + 1 \to C$ be an increasing continuous enumeration of C. Define $H : \eta + 1 \to S \cap (\eta + 1)$ by $H \restriction \xi + 1 = F$ and for any $\alpha \leq \eta, H(\xi+1+\alpha) = \pi(\alpha)$. Note that H is increasing, continuous, $H(\eta) = \eta$ and H extends g. $\square$

Theorem 2.3.25 *P_S^B preserves ω_1.*

Proof It suffices to show that if $f \in P_S^B, X \prec H_{\omega_3}, |X| = \omega, \{f, S\} \subseteq X$ and $X \cap \omega_1 \in S$, then there exists $g \leq f$ such that $g \Vdash$ "$\dot{G} \cap X$ is $P_S^B \cap X$-generic over X". Note that since $S \in X, P_S^B \in X$. Let $\alpha = X \cap \omega_1$.

Claim

$$f \cup \{(\alpha, \alpha)\} \in P_S^B.$$

Proof Let $\beta = \max(dom(f))$ and $\gamma = \max(ran(f))$. Since $f \in X, \beta < \alpha$ and $\gamma < \alpha$. Since $f \in P_S^B$, there exists a closed set $C \subseteq S$ such that $o.t.(C) = \beta+1$ and for any $\gamma \in dom(f), f(\gamma)$ is the γ-th element of C. Since $f(\beta) = \gamma$, γ is the β-th element of C. Note that for any $\beta < \alpha, \omega^\beta < \alpha$. So

α is indecomposable. Since $S \setminus (\gamma+1)$ is stationary, $S \setminus (\gamma+1) \in X$ and $\alpha \in S \setminus (\gamma+1)$, by Lemma 2.3.22 there exists a closed set $D \subseteq S \setminus (\gamma+1)$ such that $o.t.(D) = \alpha+1$ and α is the α-th element of D. Let $C^* = C \cup D$. Since α is indecomposable, $o.t.(C^*) = (\beta+1)+\alpha+1 = ((\beta+1)+\alpha)+1 = \alpha+1$. Since for any $\gamma \in dom(f)$, $f(\gamma)$ is the γ-th element of C, $\gamma = \max(ran(f))$ and $D \subseteq S \setminus (\gamma+1)$, we have for any $\gamma \in dom(f)$, $f(\gamma)$ is the γ-th element of C^*. Since α is the α-th element of D and α is indecomposable, α is the α-th element of C^*. So $f \cup \{(\alpha,\alpha)\} \in P_S^B$. $\square$

Claim $f \cup \{(\alpha,\alpha)\} \Vdash$ "$\dot{G} \cap X$ is $P_S^B \cap X$-generic over X".

Proof Let $f^* = f \cup \{(\alpha,\alpha)\}$ and $g = \{(\alpha,\alpha)\}$. Since $\alpha \in S$ and $f \cup \{(\alpha,\alpha)\} \in P_S^B$, we have $g \in P_S^B$. By Lemma 2.3.19, $g \Vdash$ "$\dot{G} \cap (P_S^B)_g$ is $(P_S^B)_g$-generic over V". So $g \Vdash$ "$\dot{G} \cap (P_S^B)_g$ is $(P_S^B)_g$-generic over X". Note that $P_S^B \cap X = P_S^B \restriction \alpha$. Since α is indecomposable, by Lemma 2.3.24, we have $(P_S^B)_g \cong P_S^B \cap X$. So $g \Vdash$ "$\dot{G} \cap X$ is $P_S^B \cap X$-generic over X". Since $f^* \in P_S^B$ and $f^* \leq g$, $f^* \Vdash$ "$\dot{G} \cap X$ is $P_S^B \cap X$-generic over X". $\square$

$\square$

Note that $|P_S^B| = \omega_1$ even not assuming CH. Since P_S^B is ω_2-$c.c$ and preserves ω_1, P_S^B preserves all cardinals.

Lemma 2.3.26 (Z_3) *P_S^B adds only Cohen reals.*

Proof Let τ be the term for a new real in the forcing language. Define

$$R_\tau = \{(1, i, f) \mid i \in \omega \wedge f \in P_S^B \wedge f \Vdash i \in \tau\} \cup \{(0, i, f) \mid i \in \omega \wedge f \in P_S^B \wedge f \Vdash i \notin \tau\}.$$

Take $X \prec H_{\omega_1}$ such that $|X| = \omega$ and $X \cap \omega_1 \in S$. Such X exists since S is stationary. Let $\eta = X \cap \omega_1$ and $f = \{(\eta, \eta)\}$. Suppose G is P_S^B-generic over V and $f \in G$. We show that τ^G is a Cohen real.

Note that $P_S^B \cap X = P_S^B \upharpoonright \eta$. By Lemma 2.3.24, we have $P_S^B \cap X \cong (P_S^B)_f$. Since $f \in P_S^B$, by Lemma 2.3.19, $f \Vdash \dot{G} \cap X$ is $P_S^B \cap X$-generic over V. So $G \cap X$ is $P_S^B \cap X$-generic over V. Note that $\tau^G = \{i \in \omega \mid \exists f \in G \cap X((1, i, f) \in R_\tau)\}$. So $\tau^G \in V[G \cap X]$. Since $P_S^B \cap X$ is countable, τ^G is a Cohen real. □

Theorem 2.3.27 (Z_3) *P_S^B preserves ω_1.*

Proof Suppose G is P_S^B-generic over V and $V[G] \models \omega_1^V$ is countable. Let x be the real which codes ω_1^V in $V[G]$. Then $V[G] \models x$ is not a Cohen real. By Lemma 2.3.26, P_S^B preserves ω_1. □

Lemma 2.3.28 *P_S^B is proper if and only if $\omega_1 \setminus S$ is not stationary.*

Proof ($\Rightarrow$) Suppose $\omega_1 \setminus S$ is stationary. Let G be P_S^B-generic over V. Then in $V[G]$, there exists a club $C \subseteq S$ and so $\omega_1 \setminus S$ is not stationary. Hence the stationarity of $\omega_1 \setminus S$ is destroyed in any generic extension of P_S^B.

($\Leftarrow$) Suppose $\omega_1 \setminus S$ is not stationary. First we show that if $X \prec H_{\omega_3}, |X| = \omega$ and $S \in X$, then $X \cap \omega_1 \in S$. Suppose $X \prec H_{\omega_3}, |X| = \omega$ and $S \in X$. Since $\omega_1 \setminus S$ is not stationary, there exists a club C on ω_1 such that $C \subseteq S$. Since $S \in X$, we can take such C in X. Since C is unbounded in $X \cap \omega_1, X \cap \omega_1 \in C$. So $X \cap \omega_1 \in S$. In Theorem 2.3.25, we have shown that if $f \in P_S^B, X \prec H_{\omega_3}, |X| = \omega, \{f, S\} \subseteq X$ and $X \cap \omega_1 \in S$, then there exists $g \leq f$ such that $g \Vdash$ "$\dot{G} \cap X$ is $P_S^B \cap X$-generic over X". Since $\{X \mid X \prec H_{\omega_3} \wedge |X| = \omega \wedge S \in X\}$ contains a club subset of $[H_{\omega_3}]^\omega$, we have $\{X \mid X \prec H_{\omega_3}, |X| = \omega, S \in X$ and if $f \in P_S^B \cap X$, then $\exists g \in P_S^B(g \leq f \wedge g \Vdash$ "$\dot{G} \cap X$ is $P_S^B \cap X$-generic over X")$\}$ contains a club subset of $[H_{\omega_3}]^\omega$. So P_S^B is proper. $\square$

Suppose G is P_S^B-generic over V and $C \subseteq S$ is the new club in $V[G]$. Then for cofinally many $\alpha < \omega_1, C \cap \alpha \notin V$ and $C \cap \alpha$ codes a Cohen real. So P_S^B is not ω_1-distributive.

Corollary 2.3.29 *P_S^B has the following properties:*

(i) P_S^B is not proper.

(ii) P_S^B is not countably closed.

(iii) P_S^B is not ω_1-distributive and adds new reals.

(iv) P_S^B adds only Cohen reals.

(v) P_S^B *preserves* ω_1.

2.3.3 The structure of the proof

We want to prove that Z_3+ **Harrington's** $\star$ does not imply "$0^\sharp$ exists". The general frame of the proof is similar as the proof of "Z_2+ **Harrington's** $\star$ does not imply $0^\sharp$ exists". To make the proof complete, we repeat and follow the frame as in the proof of Theorem 2.2.2.

We assume that Z_3+ **Harrington's** $\star$ is consistent and suppose $(M,E)\models$ Z_3+ **Harrington's** $\star$. If $(M,E)\models$ "$0^\sharp$ does not exist", we are done. Let us assume that $(M,E)\models$ "$0^\sharp$ exists". We show that "Z_3+ **Harrington's** $\star$ $+0^\sharp$ does not exist" is consistent. We work in (M,E). Since $0^\sharp$ exists, $\exists x\in\omega^\omega\,\exists\alpha(L_\alpha[x]\models Z_2+0^\sharp$ exists). Suppose δ^* is the least ordinal such that $\exists x\in\omega^\omega(L_{\delta^*}[x]\models Z_2+0^\sharp$ exists). Fix such least δ^* and some real z^* such that $L_{\delta^*}[z^*]\models Z_2+0^\sharp$ exists. Note that $\delta^*<\omega_1$ and in fact $\delta^*<\omega_1^{L[z^*]}$ since δ^* defined in V is the same as defined in $L[z^*]$. Now we work in $L_{\delta^*}[z^*]$ and hence "$Z_2+0^\sharp$ exists" holds.

Goal Find a real x such that $L_\alpha[x]\models$ "Z_3+ **Harrington's** $\star+0^\sharp$ does not exist" for some ordinal $\alpha<\delta^*$.

We find such a real x by forcing over L to get x such that $L_\alpha[x]\models$ "Z_3+ **Harrington's** $\star+0^\sharp$ does not exist" for some ordinal $\alpha<\delta^*$.

To achieve this goal, we do in five steps.[15]

Step One Force over L to get G such that $L[G] \models$ "for any L-cardinal γ, γ has weakly reflecting property".

Step Two Force over $L[G]$ to get $L[G][H]$ such that $L[G][H] \models$ "for any L-cardinal $\gamma \leq \omega_2, \gamma$ has strong reflecting property".

Step Three Force over $L[G][H]$ to get $A \subseteq \omega_1$ such that $L[G][H][A] \models$ "if $\omega_1 \leq \beta < \alpha_A$ is A-admissible, then β is an L-cardinal which has strong reflecting property" where α_A is the least α such that $L_\alpha[A] \models Z_3$.

Step Four In $L[G][H][A]$, define S as follows:

$S = \{\delta < \omega_1 \mid \exists\alpha(\alpha > \delta \wedge L_\alpha[A\cap\delta] \models Z_3 \wedge \delta = \omega_1^{L_\alpha[A\cap\delta]} \wedge \forall\eta((\delta \leq \eta < \alpha \wedge \eta$ is $A\cap\delta$-admissible$)\rightarrow \eta$ is an L-cardinal$)) \wedge \delta$ is an L-cardinal$\}$.

[15] Motivation of the proof: Do iterative Levy collapsing over L to collapse two inaccessible cardinals κ_0, κ_1 in L such that if G is $Col(\omega, <\kappa_0) * Col(\kappa_0, <\kappa_1)$-generic over L, then $L[G] \models$ "K is a club on ω_2 and for any $\gamma \in K, \gamma$ has weakly reflecting property" where $K = \{\gamma \mid \kappa_0 \leq \gamma < \kappa_1 \wedge \gamma$ is an L-cardinal$\}$. In $L[G]$, for any $\gamma \in K$, there exists a stationary subset S_γ of ω_1 by the weakly reflecting property of γ. Let P be the ω_1-product of $\{P_\gamma : \gamma \in K\}$ where P_γ is the standard Harrington-forcing to shoot a club in S_γ. In $L[G]$ force over P such that any $\alpha \in K$ has reflecting property in $L[G][H]$ where H is P-generic over $L[G]$. Do a two steps almost disjoint forcing over $L[G][H]$ to get $A \subseteq \omega_1$ such that $L[G][H][A] \models$ "if $\omega_1 \leq \beta < \alpha_A$ is A-admissible, then β is an L-cardinal which has strong reflecting property" where α_A is the least α such that $L_\alpha[A] \models Z_3$. In $L[G][H][A]$, we can show that $S = \{\delta < \omega_1 \mid \exists\alpha(\alpha > \delta \wedge L_\alpha[A\cap\delta] \models$ "$Z_3 + \delta = \omega_1$" $\wedge \forall\eta((\delta \leq \eta < \alpha \wedge \eta$ is $A\cap\delta$-admissible$)\rightarrow \eta$ is an L-cardinal$)) \wedge \delta$ is an L-cardinal$\}$ is stationary on ω_1. For any $\eta \in S$, let α_η be the least $\alpha > \eta$ such that $L_\alpha[A\cap\eta] \models$ "$Z_3 + \eta = \omega_1$". By Baumgartner's forcing, we force a club $C \subseteq S$. We show that for any $\eta \in D, L_{\alpha_\eta}[A\cap\eta, C\cap\eta] \models \eta = \omega_1$ if and only if $L_{\alpha_\eta}[A\cap\eta, C\cap\eta] \models Z_3$ where $D = \{\alpha \mid \alpha$ is a limit point of $C\}$. Let η^* be the least $\eta \in D$ such that $L_{\alpha_\eta}[A\cap\eta, C\cap\eta] \models$ "$Z_3 + \eta = \omega_1$" and we work in $L_{\alpha_{\eta^*}}[A\cap\eta^*, C\cap\eta^*]$. We define an almost disjoint system $\langle\delta_\beta : \beta < \eta^*\rangle$ on ω and $B \subseteq \eta^*$ in $L_{\alpha_{\eta^*}}[A\cap\eta^*, C\cap\eta^*]$ and then do almost disjoint forcing in $L_{\alpha_{\eta^*}}[A\cap\eta^*, C\cap\eta^*]$ to build a generic real x to code B relative to $\langle\delta_\beta : \beta < \eta^*\rangle$. Finally, we show that $L_{\alpha_{\eta^*}}[x] \models$ "Z_3 + **Harrington's** $\star$ + $0^\sharp$ does not exist".

S is stationary. For any $\eta \in S$, let α_η be the least $\alpha > \eta$ such that $L_\alpha[A \cap \eta] \models$ "$Z_3 + \eta = \omega_1$". Do Baumgartner's forcing over $L[G][H][A]$ to get a club $C \subseteq S$ on ω_1. Let $D = \{\alpha < \omega_1 \mid \alpha$ is a limit point of $C\}$. We will show that D has the following property:

For any $\eta \in D$, $L_{\alpha_\eta}[A \cap \eta, C \cap \eta] \models \eta = \omega_1$ if and only if

$L_{\alpha_\eta}[A \cap \eta, C \cap \eta] \models Z_3$.

Step Five Let η^* be the least $\eta \in D$ such that $L_{\alpha_\eta}[A \cap \eta, C \cap \eta] \models Z_3$. Work in $L_{\alpha_{\eta^*}}[A \cap \eta^*, C \cap \eta^*]$. Define a sequence $\langle \delta_\beta : \beta < \eta^* \rangle$ of almost disjoint reals and $B \subseteq \eta^*$. Do almost disjoint forcing over $L_{\alpha_{\eta^*}}[A \cap \eta^*, C \cap \eta^*]$ relative to $\langle \delta_\beta : \beta < \eta^* \rangle$ to get a real x which codes B.

Suppose $\lambda < \alpha_{\eta^*}$ and λ is x-admissible. Define

$$\theta = \sup(\{\beta < \eta^* \mid z_\beta \in L_\lambda[x]\}) \text{ and } \gamma = \sup(\{\eta_0^\beta \mid \beta < \theta\}).$$

Case 1 $\theta = \lambda$. We shall show that $\lambda = \gamma \in D$ and hence λ is an L-cardinal.

Case 2 $\theta < \lambda$.

Subcase 1 $\alpha_\gamma \leq \lambda$. We shall prove that $z_\gamma \in L_\lambda[x]$ which leads to a contradiction.

Subcase 2 $\lambda < \alpha_\gamma$. We shall show that λ is an L-cardinal from the definition of S.

Finally, we can show that $L_{\alpha_{\eta^*}}[x] \models$ "$Z_3 +$ **Harrington's** $\star$ $+ 0^\sharp$ does not exist".

Convention Throughout the following proof, we work in $L_{\delta^*}[z^*]$ in which "Z_2 $+ 0^\sharp$ exists" holds.

2.3.4 Step One

In this step we get G such that $L[G] \models$ "for any L-cardinal γ, γ has weakly reflecting property".

Definition 2.3.30 *We say that (G, γ, X) has property ∇ if and only if*

(a) $X \prec L_\gamma[G], G \in X$ and $|X| = \omega$,

(b) $\forall\lambda \in X$(if λ is an L-cardinal, then $\bar{\lambda}$ is an L-cardinal) where $\bar{\lambda}$ is the image of λ under the transitive collapse of X.

Theorem 2.3.31 *There exist $\kappa_0 < \kappa_1$ such that*

(1) κ_0, κ_1 are inaccessible in L;

(2) if G is $Col(\omega, < \kappa_0) * Col(\kappa_0, < \kappa_1)$-generic over L, then $L[G] \models$ "for any cardinal $\gamma > \kappa_1, \exists X((G, \gamma, X)$ has property $\nabla)$".

Proof Since $0^\sharp$ exists, let $j : L \to L$ be the nontrivial elementary embedding witnessed by $0^\sharp$. Let $crit(j) = \kappa, \kappa_0 = j(\kappa)$ and $\kappa_1 = j(\kappa_0)$.

Fact 2.3.32 *([9])*

(1) If $j : L \to L$ is elementary, then $crit(j)$ is a Silver indiscernible.

(2) If $0^\sharp$ exists, then any Silver indiscernible is inaccessible in L.

So κ_0 and κ_1 are inaccessible in L.

Fact 2.3.33 *([9], [12],[11]) If G is $Col(\omega, < \kappa)$-generic, then $G \cap Col(\omega, < \kappa')$ is $Col(\omega, < \kappa')$-generic for any limit ordinal $\kappa' < \kappa$.*

Lemma 2.3.34 *Suppose G is $Col(\omega, < \kappa_0)$-generic over L. Then $L[G] \models$ "for any cardinal $\gamma > \kappa_0, \exists X((G, \gamma, X)$ has property $\nabla)$".*

Proof Let $G_0 = G \cap Col(\omega, < \kappa)$. Then G_0 is $Col(\omega, < \kappa)$-generic over L.

Proposition 2.3.35 *j lifts to an elementary embedding $j^* : L[G_0] \to L[G]$ such that $j^*(G_0) = G$ and $j^* \upharpoonright L = j$.*

Proof Given $a \in L[G_0], a = \dot{x}^{G_0}$ where $\dot{x}$ is a term for a in $L^{Col(\omega,<\kappa)}$. By elementarility, $j(\dot{x})$ is a term for $j(a)$ in $L^{Col(\omega,<\kappa_0)}$ since $j(\kappa) = \kappa_0$. Let $j^*(a) = j(\dot{x})^G$.

Claim j^* is well defined.

Proof If $a = \dot{x}^{G_0} = \dot{y}^{G_0}$, then there is $p \in G_0$ such that $p \Vdash \dot{x} = \dot{y}$. By definability of the forcing relation $p \Vdash \dot{x} = \dot{y}$, applying j we get $j(p) \Vdash$

$j(\dot{x}) = j(\dot{y})$. Since $p \in G_0 \subseteq Coll(\omega, < \kappa) \subseteq L_\kappa$ and $crit(j) = \kappa$, we have $j(p) = p \in G_0 \subseteq G$. So $j(\dot{x})^G = j(\dot{y})^G$. i.e. j^* is well defined. $\square$

j^* is elementary. Let φ be a formula such that $L[G_0] \models \varphi(x)$.[16] Let $\dot{x}$ be the term for x in $L^{Col(\omega,<\kappa)}$ such that $\dot{x}^{G_0} = x$. So there is $p \in G_0$ such that $p \Vdash \varphi(\dot{x})$. Applying j we get $j(p) = p \Vdash \varphi(j(\dot{x}))$. Since $p \in G, L[G] \models \varphi(j^*(x))$. Since φ is arbitrary, j^* is elementary.

$j^* \restriction L = L$. If $a \in L$, then $a = \check{a}^{G_0}$. So $j^*(a) = j(\check{a})^G = \check{j(a)}^G = j(a)$.

$j^*(G_0) = G$. Let $\Gamma = \{\langle \check{p}, p\rangle \mid p \in Col(\omega, < \kappa)\}$. Since $\Gamma^{G_0} = G_0, j^*(G_0) = j(\Gamma)^G$ where $j(\Gamma) = \{\langle \check{p}, p\rangle \mid p \in Col(\omega, < \kappa_0)\}$. So $j^*(G_0) = G$. $\square$

Now we prove that $L[G] \models$ "for any cardinal $\gamma > \kappa_0, \exists X((G, \gamma, X)$ has property ∇)". Suppose not. Let γ be the least such counterexample. Since γ is definable in $L[G]$ from $G, \gamma \in ran(j^*)$. Let γ_0 be such that $j^*(\gamma_0) = \gamma$. Since $j^* \restriction L = j, j(\gamma_0) = \gamma$. Let

$$X = \{j^*(a) \mid a \in L_{\gamma_0}[G_0]\}.$$

Since $j^*(\gamma_0) = \gamma$ and $j^*(G_0) = G$, we have $j^*(L_{\gamma_0}[G_0]) = L_\gamma[G]$. So $X \prec L_\gamma[G]$. $G \in X$ since $G_0 \in L_{\gamma_0}[G_0]$ and $j^*(G_0) = G$.

Suppose $\lambda \in X$ is an L-cardinal and $\lambda = j^*(\bar{\lambda})$. Then $\bar{\lambda}$ is an L-cardinal. Let M be the transitive collapse of X and $\pi : X \cong M$ be the collapsing map. Since $j^* \restriction L_{\gamma_0}[G_0] : L_{\gamma_0}[G_0] \cong X$, $\pi \circ (j^* \restriction L_{\gamma_0}[G_0]) : L_{\gamma_0}[G_0] \cong M$. Since

[16] For simplicity, we suppose that φ has only one free variable.

$(\pi \circ j^*)(\bar{\lambda}) = \pi(\lambda)$ and $\bar{\lambda}$ is an L-cardinal, $\pi(\lambda)$ is an L-cardinal.

Note that γ is definable in L from κ_0. Since $0^\sharp$ exists, γ is less than the least indiscernible above κ_0. Since $j(\gamma_0) = \gamma, \gamma_0 < \kappa_0$. So γ_0 is countable in $L[G]$ since G is $Col(\omega, < \kappa_0)$-generic over L. Since $G, j^* \in V[G]$, we have $X \in V[G]$. Since γ_0 is countable in $L[G]$, X is countable in $V[G]$. If $X \in L[G]$, then X is countable in $L[G]$ since γ_0 is countable in $L[G]$.

Claim

$$X \notin L[G].$$

Proof If $X \in L[G]$, then $j \restriction L_{\gamma_0} \in L[G]$.

Fact 2.3.36 *([9]) Let $j : L_\alpha \to L_\beta$ be an elementary embedding and $crit(j) = \gamma$. If $\gamma < |\alpha|$, then $0^\sharp$ exists.*

Since $j(\gamma_0) = \gamma, j \restriction L_{\gamma_0} : L_{\gamma_0} \to L_\gamma$. Since $\gamma > \kappa_0 > \kappa = crit(j), \gamma_0 > \kappa$. So $crit(j \restriction L_{\gamma_0}) = \kappa$. So if $X \in L[G], L[G] \models 0^\sharp$ exists. Contradiction. $\square$

Since $X \notin L[G]$, we can not claim that $L[G] \models \exists X((G, \gamma, X)$ has property $\nabla)$. We only have that $V[G] \models \exists X((G, \gamma, X)$ has property $\nabla)$. We build a tree T such that the infinite branch of T corresponds to a witness X' for property ∇ whose transitive collapse is the same as the transitive collapse of X. Let $\bar{X}$ be the transitive collapse of X. Note that $\bar{X} \in L[G]$. Let $T = \{e \restriction n : n \in \omega, e : \omega \to L_\gamma[G], e``\omega \prec L_\gamma[G], G \in e``\omega$, the transitive collapse of

$e``\omega$ is $\bar{X}$ and $\forall\lambda \in e``\omega$(if λ is an L-cardinal, then the image of λ under the transitive collapse of $e``\omega$ is an L-cardinal)$\}$. T is a tree. Since $\bar{X} \in L[G]$, from the definition of $T, T \in L[G]$. Since X is a witness in $V[G]$ such that (G, γ, X) has property ∇, T has an infinite branch in $V[G]$. By absoluteness, T must have an infinite branch in $L[G]$ which corresponds to a witness in $L[G]$ for property ∇. So we have $L[G] \models \exists X((G, \gamma, X)$ has property $\nabla)$. But since γ is the counterexample, we have $L[G] \not\models \exists X((G, \gamma, X)$ has property $\nabla)$. Contradiction. $\square$

Suppose G is $Col(\omega, < \kappa_0) * Col(\kappa_0, < \kappa_1)$-generic over L. Let $G_0 = G \cap Col(\omega, < \kappa_0)$. Then G_0 is $Col(\omega, < \kappa_0)$-generic over L. Similarly as in the proof of Lemma 2.3.34, we can build a lift embedding $j^* : L[G_0] \to L[G]$ such that $j^*(G_0) = G$ and $j^* \upharpoonright L = j$. Note that $L[G] \models$ "for any cardinal $\gamma > \kappa_1, \exists X((G, \gamma, X)$ has property $\nabla)$" if and only if $L[G_0] \models$ "for any cardinal $\gamma > \kappa_0, \exists X((G_0, \gamma, X)$ has property $\nabla)$". So Theorem 2.3.31 follows from Lemma 2.3.34. $\square$

Fix κ_0 and κ_1 as defined in Theorem 2.3.31. In $L, \{\alpha \mid \kappa_0 \leq \alpha < \kappa_1 \wedge \alpha$ is an L-cardinal$\}$ is a club on κ_1. Let G be $Col(\omega, < \kappa_0) * Col(\kappa_0, < \kappa_1)$-generic over L. Now we work in $L[G]$.

Define

$$K = \{\gamma \mid \omega_1 \leq \gamma < \omega_2 \wedge \gamma \text{ is an } L\text{-cardinal}\}.$$

In $L[G], \kappa_1^L = \omega_2, \kappa_0^L = \omega_1$, K is a club on ω_2 and $K \cap \omega_1 = \emptyset$.

Corollary 2.3.37 $L[G] \models$ *"for any L-cardinal γ, γ has weakly reflecting property".*

Proof We work in $L[G]$. Assume that there is an L-cardinal which does not have weakly reflecting property. Let α be the least such L-cardinal. Thus α is definable in $L_\gamma[G]$ from G for all sufficiently large cardinal γ. Fix a large enough cardinal $\gamma > \kappa_1$. By Theorem 2.3.31, there exists X such that (G, γ, X) has property ∇. So $X \prec L_\gamma[G], |X| = \omega$ and $G \in X$. Since α is definable in $L_\gamma[G]$ from G, we have $\alpha \in X$. Since $\alpha \in X$ is an L-cardinal, by the property of ∇, $\bar{\alpha}$ is an L-cardinal where $\bar{\alpha}$ is the image of α under the transitive collapse of X. By Proposition 2.3.8, α has weakly reflecting property which leads to a contradiction. $\Box$

Especially, $L[G] \models$ "K is a club on ω_2 and for any $\gamma \in K, \gamma$ has weakly reflecting property".

2.3.5 Step Two

In this step we force over $L[G]$ to get $L[G][H]$ such that $L[G][H] \models$ "for any L-cardinal $\gamma \leq \omega_2, \gamma$ has strong reflecting property".

Now we work in $L[G]$. For any $\gamma \in K$, since γ has weakly reflecting property, there exist a bijection $\pi : \omega_1 \leftrightarrow \gamma$ and a stationary set $S \subseteq \omega_1$ such that for any $\theta \in S, o.t.(\{\pi(\alpha) \mid \alpha < \theta\})$ is an L-cardinal and let π_γ and S_γ be such π and S. Then for any $\gamma \in K, S_\gamma$ is stationary.

Definition 2.3.38 *Suppose κ is a regular cardinal and $\{P_i : i \in I\}$ is a collection of partially ordered sets. The κ-product of $\{P_i : i \in I\}$ is defined as*

$$P = \{p : dom(p) = I \wedge \forall i \in I(p(i) \in P_i) \wedge |s(p)| < \kappa\}$$

where $s(p) = \{i \in I : p(i) \neq 1_{P_i}\}$.

Let P_γ be forcing notion shooting a club in S_γ defined in section 2.1.2. Let P be the ω_1-product of $\{P_\gamma : \gamma \in K\}$. Since CH holds in $L[G], |P_\gamma| = \omega_1$ for any $\gamma \in K$.

Fact 2.3.39 *([9]) Assume $\kappa^{<\kappa} = \kappa$. If for every $i \in I, |P_i| \leq \kappa$, then the κ-product of P_i satisfies κ^+-c.c.*

In $L[G], \omega_1^{<\omega_1} = \omega_1$. So P has ω_2-c.c. For any $\gamma \in K, P_\gamma$ is ω_1-distributive and hence preserves ω_1.

Lemma 2.3.40 *P is ω_1-distributive.*

Proof It suffices to show that if $p \Vdash \dot{f} : \omega \to Ord$, then $\exists q \leq p \exists g(q \Vdash \dot{f} = \check{g})$. Suppose $p \Vdash \dot{f} : \omega \to Ord$. By induction on α we construct a chain $\{A_\alpha : \alpha < \omega_1\}$ of countable subsets of P.

Let $A_0 = \{p\}$. If α is limit, let $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$. For $\gamma \in K$ and $\alpha < \omega_1$, define

$$\theta_\alpha^\gamma = sup(\{sup(q(\gamma)) : q \in A_\alpha\}).$$

Given A_α we define $A_{\alpha+1}$ as follows. Note that for any $p \in A_\alpha$ and $n \in \omega$, there is $q = q(p, n) \in P$ such that

$$q \leq p, q \text{ decides } \dot{f}(n) \text{ and } \forall \gamma \in s(q)(sup(q(\gamma)) > \theta_\alpha^\gamma).$$

Let $A_{\alpha+1} = A_\alpha \cup \{q(p, n) \mid p \in A_\alpha, n \in \omega\}$. For $\gamma \in K$ and $\alpha < \omega_1$, since A_α is countable and $q(\gamma) \in P_\gamma$ is a bounded subset of ω_1, we have $\theta_\alpha^\gamma < \omega_1$. From our definition of θ_α^γ and A_α, for any $\gamma \in K$, $\langle \theta_\alpha^\gamma : \alpha < \omega_1 \rangle$ is increasing and continuous. For any $\gamma \in K$, let

$$C_\gamma = \{\eta < \omega_1 : \alpha < \eta \rightarrow \theta_\alpha^\gamma < \eta\}.$$

For any $\gamma \in K, C_\gamma$ is a club on ω_1. Since S_γ is stationary, there is $\eta_\gamma \in S_\gamma \cap C_\gamma$ such that η_γ is a limit point of C_γ. Given $\gamma \in K$, take $\{\alpha_n^\gamma : n \in \omega\} \subseteq C_\gamma$ such that $\lim_{n\in\omega} \alpha_n^\gamma = \eta_\gamma$. Then $\lim_{n\in\omega} \theta_{\alpha_n^\gamma}^\gamma = \eta_\gamma$.

Now we construct a sequence $\langle p_n : n \in \omega \rangle$ by induction.

Let $p_0 = p$ and $s_0 = s(p_0)$. Let $\beta_1 = \min\{\alpha_1^\gamma : \gamma \in s_0\}$. Take $p_1 \in A_{\beta_1}$ such that $p_1 \leq p_0$ and p_1 decides $\dot{f}(0)$. Let $s_1 = s(p_1)$. Suppose we have defined p_n, β_n and s_n. Let $\beta_{n+1} = \min\{\alpha_{n+1}^\gamma : \gamma \in s_n\}$. Take $p_{n+1} \in A_{\beta_{n+1}}$ such that $p_{n+1} \leq p_n$ and p_{n+1} decides $\dot{f}(n)$. Then for any $\gamma \in s_n, p_{n+1} \in A_{\alpha_{n+1}^\gamma}$. Without loss of generality, we can take $\langle \beta_n : n \in \omega \rangle$ to be increasing since we only need that there are enough $\gamma \in s_n$ such that $p_{n+1} \in A_{\alpha^\gamma_{n+1}}$. Let $s_{n+1} = s(p_{n+1})$. Continue this process ω times.

Let $s = \bigcup_n s_n$. Since each s_n is countable, s is at most countable. Note that for any $\gamma \in s$ there is $N \in \omega$ such that for all $n \geq N, p_{n+1} \in A_{\alpha^\gamma_{n+1}}$. So

$$\forall \gamma \in s\, \exists N\, \forall n \geq N \left(\theta^\gamma_{\alpha^\gamma_n} < sup(p_{n+1}(\gamma)) \leq \theta^\gamma_{\alpha^\gamma_{n+1}} \right).$$

Hence for any $\gamma \in s, \lim_{n\in\omega} sup(p_n(\gamma)) = \eta_\gamma$ since $\lim_{n\in\omega} \theta^\gamma_{\alpha^\gamma_n} = \eta_\gamma$.

Now we define the q we want as: if $\gamma \in s$, then $q(\gamma) = \bigcup_n p_n(\gamma) \cup \{\eta_\gamma\}$; otherwise, let $q(\gamma) = 1_{P_\gamma}$. For any $\gamma \in s$, since $\eta_\gamma \in S_\gamma, q(\gamma)$ is a closed bounded subset of S_γ. So for any $\gamma \in s, q(\gamma) \in P_\gamma$. Hence $q \in P$. Since $\forall n \in \omega(q \leq p_n \wedge p_{n+1}$ decides $\dot{f}(n))$, q decides $\dot{f}(n)$ for any $n \in \omega$. Define g as: $g(n) = \dot{f}(n)$. Then $q \leq p$ and $q \Vdash \dot{f} = \check{g}$. □

So P preserves ω_1 and hence adds no new reals. Hence P preserves all cardinals.

Remark Generally, distributivity is not preserved by products. For example, the product of infinitely many perfect set forcing with finite support collapses ω_1.

Let H be P-generic over $L[G]$. In $L[G][H]$, K is a club on ω_2 and for any $\alpha \in K, \alpha$ has reflecting property. In $L[G][H]$, since K is a club on ω_2, ω_2 is a limit cardinal in L. By Proposition 2.3.11 and Proposition 2.3.14, the following hold in $L[G][H]$:

- Any $\alpha \in K$ has strong reflecting property.

- ω_2 has strong reflecting property.

In fact, this is the best we can prove in $L[G][H]$ since by Theorem 2.3.15, if $L[G][H] \models$ "there exists an L-cardinal $\gamma > \omega_2$ such that γ has strong reflecting property", then $L[G][H] \models$ "$0^\sharp$ exists".

2.3.6 Step Three

In this step we force over $L[G][H]$ to get $A \subseteq \omega_1$ such that $L[G][H][A] \models$ "if $\omega_1 \leq \beta < \alpha_A$ is A-admissible, then β is an L-cardinal which has strong reflecting property" where α_A is the least α such that $L_\alpha[A] \models Z_3$.

Now we work in $L[G, H]$. We know that $K = \{\omega_1 \leq \gamma < \omega_2 \mid \gamma$ is an L-cardinal$\}$ is a club on ω_2 and any element of K has strong reflecting property. Note that GCH holds in $L[G, H]$. Take a $B \subseteq \omega_2$ such that (1)$\omega^\omega \subseteq L[B]$; (2) $K \in L[B]$; (3) $\omega_2^{L[B]} = \omega_2$ and (4) $(L_{\omega_2}[B], K) \prec (H_{\omega_2}, K)$.

Now we work in $L[B]$. To define an almost disjoint sequence $\langle \delta^*_\beta \mid \beta < \omega_2\rangle$ of subsets of ω_1, we firstly define a sequence $\langle \sigma^*_\beta \mid \beta < \omega_2\rangle$ of distinct subsets of ω_1. Let σ^*_0 be the $<_{L[B]}$-least subset of ω_1. Fix $\gamma < \omega_2$. Suppose we have defined $\langle \sigma^*_\beta \mid \beta < \gamma\rangle$. Since $L[B] \models \gamma < \omega_2 \wedge 2^{\omega_1} = \omega_2$, let σ^*_γ be the $<_{L[B]}$-least subset of ω_1 which is different from σ^*_β for any $\beta < \gamma$. From $\langle \sigma^*_\beta \mid \beta < \omega_2\rangle$, we can define an almost disjoint sequence $\langle \delta^*_\beta \mid \beta < \omega_2\rangle$ as follows. Let $\langle s_\alpha \mid \alpha \in \omega_1\rangle \in L[B]$ be a $<_{L[B]}$-least enumeration of $\omega_1^{<\omega_1}$. For any $\beta < \omega_2$, define

$$\delta^*_\beta = \{\alpha \in \omega_1 \mid \exists \eta \in \omega_1 (s_\alpha = \sigma^*_\beta \cap \eta)\}.$$

It is easy to check that $\langle \delta^*_\beta : \beta < \omega_2 \rangle$ is a sequence of almost disjoint subset of ω_1.

Now we do almost disjoint forcing to code B via $\langle \delta^*_\beta \mid \beta < \omega_2 \rangle$. Then we get $A_0 \subseteq \omega_1$ such that $\alpha \in B \Leftrightarrow |A_0 \cap \delta^*_\alpha| < \omega_1$. The almost disjoint forcing preserves all cardinals.

Now we work in $L[A_0]$. Let $C = K \cap \{\eta \mid L_\eta[A_0] \prec L_{\omega_2}[A_0]\}$. Define $F : \mathcal{P}(\omega_1) \to \mathcal{P}(\omega_1)$ as follows: if $y \subseteq \omega_1$ codes γ, then $F(y) \subseteq \omega_1$ codes $(\beta, C \cap \beta)$ where β is the least element of C such that $\beta > \gamma$ and $(L_\beta[A_0, C], C \cap \beta) \prec (L_{\omega_2}[A_0, C], C)$. Since C is a club on ω_2, in the definition of $F(y)$ such β exists.

To define an almost disjoint sequence $\langle \delta_\beta \mid \beta < \omega_2 \rangle$ on ω_1, we firstly define a sequence $\langle \sigma_\beta \mid \beta < \omega_2 \rangle$ of distinct subsets of ω_1. Let σ_0 be the $<_{L[A_0,C]}$-least subset of ω_1. Fix $\gamma < \omega_2$. Suppose we have defined $\langle \sigma_\beta \mid \beta < \gamma \rangle$. Since $L[A_0, C] \models \gamma < \omega_2 \wedge 2^{\omega_1} = \omega_2$, let σ_γ be the $<_{L[A_0,C]}$-least subset of ω_1 which is different from σ_β for any $\beta < \gamma$. From $\langle \sigma_\beta \mid \beta < \omega_2 \rangle$, we can define an almost disjoint sequence $\langle \delta_\beta \mid \beta < \omega_2 \rangle$ as follows. Let $\langle t_\alpha \mid \alpha \in \omega_1 \rangle \in L[A_0, C]$ be a $<_{L[A_0,C]}$-least enumeration of $\omega_1^{<\omega_1}$. For any $\beta < \omega_2$, define

$$\delta_\beta = \{\alpha \in \omega_1 \mid \exists \eta \in \omega_1 (t_\alpha = \sigma_\beta \cap \eta)\}.$$

It is easy to check that $\langle \delta_\beta : \beta < \omega_2 \rangle$ is a sequence of almost disjoint subset of ω_1.

Let $\langle x_\alpha \mid \alpha < \omega_2 \rangle$ be the enumeration of $\mathcal{P}(\omega_1)$ in $L[A_0, C]$ in the order of construction. Define $Z_F \subseteq \omega_2$ as follows.

$$Z_F = \{\alpha \cdot \omega + \beta \mid \alpha < \omega_2 \wedge \beta \in F(x_\alpha)\}.$$

Now we do almost disjoint forcing to code Z_F via $\langle \delta_\beta \mid \beta < \omega_2 \rangle$. Then we get $A_1 \subseteq \omega_1$ such that $\beta \in Z_F \Leftrightarrow |A_1 \cap \delta_\beta| < \omega_1$. Let $A = (A_0, A_1)$. The almost disjoint forcing preserves all cardinals.

Now we work in $L[G, H, A]$. Let α_A be the least α such that $L_\alpha[A] \models Z_3$. Note that $\omega_1 < \alpha_A < \omega_2$. Now we show that if $\omega_1 \leq \alpha < \alpha_A$ is A-admissible, then α is an L-cardinal which has strong reflecting property. Suppose $\omega_1 \leq \alpha < \alpha_A$ is A-admissible. We show that α is an L-cardinal which has strong reflecting property.

Define

$$\gamma_0 = \sup(\{\gamma < \alpha \mid (L_\gamma[A_0, C], C \cap \gamma) \prec (L_{\omega_2}[A_0, C], C)\}).$$

If there is no $\gamma < \alpha$ such that $(L_\gamma[A_0, C], C \cap \gamma) \prec (L_{\omega_2}[A_0, C], C)$, then let $\gamma_0 = 0$. Note that if $\gamma < \omega_2$ and $(L_\gamma[A_0, C], C \cap \gamma) \prec (L_{\omega_2}[A_0, C], C)$, then $\gamma \in C$. So if $\gamma_0 > 0$, then $\gamma_0 \in C$, $(L_{\gamma_0}[A_0, C], C \cap \gamma_0) \prec (L_{\omega_2}[A_0, C], C)$ and $L_{\gamma_0}[A_0] = L_{\gamma_0}[A_0, C]$. From the definition of γ_0, we have $\gamma_0 \leq \alpha$. We assume that $\gamma_0 < \alpha$ and try to get a contradiction.

Let α_0 be the least A_0-admissible ordinal such that $\alpha_0 > \gamma_0$ and $\alpha_0 \geq \omega_1$. Since α is A-admissible and $\gamma_0 < \alpha$, we have $\alpha_0 \leq \alpha$.

Claim

$$C \cap \alpha_0 = C \cap (\gamma_0 + 1).$$

Proof We show that $C \cap \alpha_0 \subseteq C \cap (\gamma_0 + 1)$. Suppose $\gamma \in C \cap \alpha_0$ and $\gamma > \gamma_0$. Since $\gamma \in C$, $L_\gamma[A_0] \prec L_{\omega_2}[A_0]$. Since α_0 is definable from γ_0 and A_0, we have α_0 is definable in $L_\gamma[A_0]$. So $\alpha_0 \leq \gamma$. Contradiction. $\square$

Since $C \cap \alpha_0 = C \cap (\gamma_0 + 1)$, we have $L_{\alpha_0}[C, A_0] = L_{\alpha_0}[C \cap \gamma_0, A_0]$.

Claim $L_{\alpha_0}[C \cap \gamma_0, A_0] \models \gamma_0 < \omega_2$.

Proof Suppose $L_{\alpha_0}[C \cap \gamma_0, A_0] \models \gamma_0 \geq \omega_2$. Let P be the partial order for almost disjoint coding Z_F via the almost disjoint system $\langle \delta_\beta \mid \beta < \omega_2 \rangle$. Note that P is definable over $L_{\omega_2+1}[A_0, C]$. Let $P^* = P \cap L_{\gamma_0}[A_0, C]$. Since $(L_{\gamma_0}[A_0, C], C \cap \gamma_0) \prec (L_{\omega_2}[A_0, C], C)$, we have P^* is definable over $L_{\gamma_0+1}[A_0, C]$ and hence $P^* \in L_{\alpha_0}[C \cap \gamma_0, A_0]$. Since P^* is ω_2-c.c in $L_{\alpha_0}[C \cap \gamma_0, A_0]$, we have $\gamma_0 = \omega_2^{L_{\alpha_0}[C \cap \gamma_0, A_0]}$.

We show that A_1 is generic over $L_{\alpha_0}[C \cap \gamma_0, A_0]$ for P^*. Let $Y \subseteq P^*$ be a maximal antichain with $Y \in L_{\alpha_0}[C \cap \gamma_0, A_0]$. Since P^* is ω_2-c.c in $L_{\alpha_0}[C \cap \gamma_0, A_0]$ and $\gamma_0 = \omega_2^{L_{\alpha_0}[C \cap \gamma_0, A_0]}$, we have $Y \in L_{\gamma_0}[C \cap \gamma_0, A_0] = L_{\gamma_0}[C, A_0] = L_{\gamma_0}[A_0]$. Since $(L_{\gamma_0}[A_0, C], C \cap \gamma_0) \prec (L_{\omega_2}[A_0, C], C)$, it follows that Y is a maximal antichain in P. So the filter given by A_1 meets Y and hence A_1 is generic over $L_{\alpha_0}[C \cap \gamma_0, A_0]$ for P^*.

Note that $L_{\alpha_0}[A_0, C\cap\gamma_0]\cap 2^{\omega_1} = L_{\gamma_0}[A_0, C\cap\gamma_0]\cap 2^{\omega_1}$. So $\gamma_0 = \omega_2^{L_{\alpha_0}[C\cap\gamma_0, A_0]} = \omega_2^{L_{\alpha_0}[C\cap\gamma_0, A_0][A_1]} = \omega_2^{L_{\gamma_0}[C\cap\gamma_0, A_0][A_1]}$. Since $L_{\gamma_0}[A_0, C \cap \gamma_0][A_1] = L_{\gamma_0}[A]$, we have $\gamma_0 = \omega_2^{L_{\gamma_0}[A]}$ and hence $L_{\gamma_0}[A] \models Z_3$. Note that $\gamma_0 < \alpha_0 < \alpha_A$. This contradicts that α_A is the least ordinal ξ such that $L_\xi[A] \models Z_3$. $\square$

Note that for any $\eta < \alpha_0, L_{\alpha_0}[C, A_0] = L_{\alpha_0}[C \cap \gamma_0, A_0] \models \eta < \omega_2$. From our definition of $\langle \delta_\beta : \beta < \omega_2\rangle$ and $\langle x_\alpha \mid \alpha < \omega_2\rangle$, we have:

(i) For each $\eta < \alpha_0, \langle \delta_\beta : \beta < \eta\rangle \in L_{\alpha_0}[C, A_0] = L_{\alpha_0}[C \cap \gamma_0, A_0]$.

(ii) For each $\eta < \alpha_0, \langle x_\beta : \beta < \eta\rangle \in L_{\alpha_0}[C, A_0] = L_{\alpha_0}[C \cap \gamma_0, A_0]$.

(iii) $\langle x_\beta \mid \beta < \alpha_0\rangle$ enumerates $\mathcal{P}(\omega_1) \cap L_{\alpha_0}[C \cap \gamma_0, A_0]$.

Claim

$$C \cap \gamma_0 \in L_{\gamma_0+1}[A].$$

Proof If $\gamma_0 = 0$, this is trivial. Suppose $\gamma_0 > 0$. Let

$$D = \{\gamma \in C \cap \alpha_A \mid (L_\gamma[A_0, C], C \cap \gamma) \prec (L_{\omega_2}[A_0, C], C)\}.$$

We prove by induction that for any $\gamma \in D, C \cap \gamma$ is definable in $L_\gamma[A]$ from A. Fix $\gamma \in D$. Suppose for any $\gamma' \in D \cap \gamma$, $C \cap \gamma' \in L_{\gamma'+1}[A]$. We show that $C \cap \gamma$ is definable in $L_\gamma[A]$ from A.

Let $\eta \geq \omega_1$ be the least ordinal such that $L_\eta[A_0, C \cap \gamma]$ is admissible. Since $\gamma \in \alpha_A, L_\eta[A_0, C \cap \gamma] \models \gamma < \omega_2$. (If not, then by the similar argument as we show that $L_{\alpha_0}[C \cap \gamma_0, A_0] \models \gamma_0 < \omega_2$, A_1 is generic over $L_\eta[A_0, C \cap \gamma]$ and so $L_\gamma[A] \models Z_3$ which leads a contradiction.)

Since $(L_\gamma[A_0, C], C \cap \gamma) \prec (L_{\omega_2}[A_0, C], C)$, we have $L_\gamma[A_0] = L_\gamma[A_0, C]$. Note that $C \cap \eta = C \cap (\gamma+1)$. Note that for any $\beta < \eta$, $L_\eta[A_0, C \cap \gamma] \models \beta < \omega_2$. From our definitions, for any $\beta < \eta$ we have:

(i) $\langle x_\xi \mid \xi \in \beta \rangle \in L_\eta[A_0, C \cap \gamma]$.

(ii) $\langle \delta_\xi \mid \xi \in \beta \rangle \in L_\eta[A_0, C \cap \gamma]$.

(iii) $\langle x_\xi \mid \xi \in \eta \rangle$ enumerates $\mathcal{P}(\omega_1) \cap L_\eta[A_0, C] = \mathcal{P}(\omega_1) \cap L_\eta[A_0, C \cap \gamma]$.

Suppose $y \subseteq \omega_1$ and $y \in L_\eta[A_0, C \cap \gamma]$. Then $y = x_\xi$ for some $\xi < \eta$. Note that $\xi \cdot \omega + \alpha < \eta$ for any $\alpha < \omega_1$. $\alpha \in F(y)$ if and only if $|A_1 \cap \delta_{\xi \cdot \omega + \alpha}| < \omega_1$. So $F(y) \in L_\eta[A_0, C \cap \gamma][A_1]$. Hence we have shown that if $y \in \mathcal{P}(\omega_1) \cap L_\eta[A_0, C \cap \gamma]$, then $F(y) \in L_\eta[A, C \cap \gamma]$.

Case 1: There exists β such that γ is the least element of C such that $\gamma > \beta$ and $(L_\gamma[A_0, C], C \cap \gamma) \prec (L_{\omega_2}[A_0, C], C)$. Since $L_\eta[A_0, C \cap \gamma] \models \gamma < \omega_2$, we have $L_\eta[A_0, C \cap \gamma] \models \beta < \omega_2$. Take $y \in L_\eta[A_0, C \cap \gamma] \cap \mathcal{P}(\omega_1)$ such that y codes β. So $F(y)$ codes $(\gamma, C \cap \gamma)$ and $F(y) \in L_\eta[A, C \cap \gamma]$. Since $\gamma \in D$ and $\eta \geq \omega_1$ is the least ordinal such that $L_\eta[A_0, C \cap \gamma]$ is admissible, we have $F(y)$

is definable in $L_\gamma[A_0, C\cap\gamma][A] = L_\gamma[A]$ from A. Since $F(y)$ codes $C\cap\gamma$, we have $C\cap\gamma$ is definable in $L_\gamma[A]$ from A. So $C\cap\gamma \in L_{\gamma+1}[A]$.

Case 2: Such β does not exist. Then γ is a limit point of $\{\gamma' \in C \mid (L_{\gamma'}[A_0, C], C\cap\gamma') \prec (L_{\omega_2}[A_0, C], C)\}$. Let $\gamma = \sup(\{\gamma_n : n \in \omega\})$ where $\gamma_n \in D$. So $C\cap\gamma_n \in L_{\gamma_n+1}[A]$ for any $n \in \omega$. Note that $C\cap\gamma = \bigcup_{n\in\omega}(C\cap\gamma_n)$. So $C\cap\gamma \in L_{\gamma+1}[A]$.

Since $\gamma_0 \in D$, we have $C\cap\gamma_0 \in L_{\gamma_0+1}[A]$. $\square$

Claim If $y \subseteq \omega_1$ and $y \in L_{\alpha_0}[C\cap\gamma_0, A_0]$, then $F(y) \in L_{\alpha_0}[A]$.

Proof Since $\langle x_\beta \mid \beta < \alpha_0\rangle$ enumerates $\mathcal{P}(\omega_1) \cap L_{\alpha_0}[C\cap\gamma_0, A_0]$, we have $y = x_\xi$ for some $\xi < \alpha_0$. Note that for any $\alpha < \omega_1, \xi\cdot\omega+\alpha < \alpha_0$. By the definition of $Z_F, \alpha \in F(y) \Leftrightarrow \alpha \in F(x_\xi) \Leftrightarrow \xi\cdot\omega+\alpha \in Z_F \Leftrightarrow |A_1 \cap \delta_{\xi\cdot\omega+\alpha}| < \omega_1$. Since for each $\eta < \alpha_0, \langle\delta_\beta : \beta < \eta\rangle \in L_{\alpha_0}[C\cap\gamma_0, A_0]$, we have $F(y) \in L_{\alpha_0}[C\cap\gamma_0, A_0][A_1]$. Since $C\cap\gamma_0 \in L_{\gamma_0+1}[A]$, we have $L_{\alpha_0}[C\cap\gamma_0, A_0] \subseteq L_{\alpha_0}[A]$. So $F(y) \in L_{\alpha_0}[A]$. $\square$

Since $L_{\alpha_0}[C\cap\gamma_0, A_0] \models \gamma_0 < \omega_2$, there exists $y \in L_{\alpha_0}[C\cap\gamma_0, A_0] \cap \mathcal{P}(\omega_1)$ such that y codes γ_0. Note that $F(y)$ codes $(\gamma_1, C\cap\gamma_1)$ where γ_1 is the least element of C such that $\gamma_1 > \gamma_0$ and $(L_{\gamma_1}[A_0, C], C\cap\gamma_1) \prec (L_{\omega_2}[A_0, C], C)$.

Since $F(y)$ codes $(\gamma_1, C\cap\gamma_1)$ and $F(y) \in L_{\alpha_0}[A]$, we have $\gamma_0 < \gamma_1 < \alpha_0 \leq \alpha$. Since $\gamma_1 < \alpha$ and $(L_{\gamma_1}[A_0, C], C\cap\gamma_1) \prec (L_{\omega_2}[A_0, C], C)$, by the definition of γ_0 we have $\gamma_1 \leq \gamma_0$. Contradiction.

So the assumption that $\gamma_0 < \alpha$ is false and we have $\gamma_0 = \alpha$. Then $(L_\alpha[A_0, C], C \cap \alpha) \prec (L_{\omega_2}[A_0, C], C)$. So $\alpha \in C$ and hence α is an L-cardinal with strong reflecting property.

So in $L[G][H]$ we get $A \subseteq \omega_1$ such that $L[G][H][A] \models$ "if $\omega_1 \leq \alpha < \alpha_A$ is A-admissible, then α is an L-cardinal with strong reflecting property" where α_A is the least ordinal such that $L_{\alpha_A}[A] \models Z_3$.

2.3.7 Step Four

Now we work in $L[G][H][A]$.

Fact 2.3.41 *([2], [9])*

(1) If $M \prec L_\gamma[A], |M| = \omega$ and $\gamma \geq \omega_1$ is limit, then $M \cap \omega_1 = \alpha$ for some $\alpha < \omega_1$.[17]

(2) Suppose $\alpha > \omega_1$ is a limit ordinal, $Y \prec L_\alpha[A]$ and $|Y| = \omega$. Let $\bar{Y}$ be the transitive collapse of Y. Then $\bar{Y} = L_{\bar{\alpha}}[\bar{A}]$ where $\bar{\omega_1} = Y \cap \omega_1 < \omega_1, \bar{A} = A \cap \bar{\omega_1}$ and $\bar{\alpha} = o.t.(Y \cap \alpha)$.

(3) "$\exists A \subseteq \omega_1(V = L[A]) + Z_3$" $\vdash \omega_1$ is the largest cardinal.[18]

Suppose $Y \prec L_{\alpha_A}[A], |Y| = \omega$ and $\bar{Y}$ is the transitive collapse of Y where α_A is the least α such that $L_\alpha[A] \models Z_3$. Let $\bar{\omega_1} = Y \cap \omega_1$. Then $\bar{Y} = L_{\bar{\alpha}}[\bar{A}]$

[17] As a corollary, if $M \prec L_{\omega_1}[A]$ and $|M| = \omega$, then M is transitive.

[18] It is a fact that if $A \subseteq \omega_1$, then $L[A] \models GCH$.

where $\bar{A} = A \cap \bar{\omega_1}$ and $\bar{\alpha} = o.t.(Y \cap \alpha_A)$. Note that $\bar{\omega_1} < \omega_1$ and $L_{\bar{\alpha}}[\bar{A}] \models Z_3$. Suppose $\bar{\omega_1} \leq \eta < \bar{\alpha}$ is $\bar{A}$-admissible. We know that if $\omega_1 \leq \alpha < \alpha_A$ is A-admissible, then α is an L-cardinal which has strong reflecting property. So η is an L-cardinal. Let

$$Z = \{\delta < \omega_1 \mid \exists\alpha(\alpha > \delta \wedge L_\alpha[A \cap \delta] \models \text{“}Z_3 + \delta = \omega_1\text{”} \wedge \forall\eta((\delta \leq \eta < \alpha \wedge \eta \text{ is } A \cap \delta\text{-admissible}) \rightarrow \eta \text{ is an } L\text{-cardinal}))\}.$$

Note that $Y \cap \omega_1 \in Z$. (In the definition of Z, replace δ and α with $\bar{\omega_1}$ and $\bar{\alpha}$ respectively. $\omega_1^{L_{\bar{\alpha}}[A \cap \bar{\omega_1}]} = \omega_1^{L_{\bar{\alpha}}[\bar{A}]} = \omega_1^{\bar{Y}} = \omega_1 \cap Y = \bar{\omega_1}$.) So we have shown that $Y \cap \omega_1 \in Z$ for any $Y \prec L_{\alpha_A}[A]$ with $|Y| = \omega$. From Fact 2.3.2, there exists a club Q on ω_1 such that $Q \subseteq Z$. We know that in $L[G][H]$, for any $\gamma \in K, \gamma$ has strong reflecting property and especially, ω_1 has strong reflecting property. So by Proposition 2.3.5, in $L[G][H]$ there exists a club N on ω_1 such that $N \subseteq \{\alpha < \omega_1 : \alpha \text{ is an } L\text{-cardinal}\}$. Also in $L[G][H][A], N$ is a club on ω_1 such that any $\alpha \in N$ is an L-cardinal.

Define $S = Z \cap \{\alpha < \omega_1 : \alpha \text{ is an } L\text{-cardinal}\}$. Note that S is definable in $(L_{\omega_1}[A], \in, A)$. Since Q, N are clubs on ω_1 and $Q \cap N \subseteq S, S$ is stationary on ω_1.

Notation For any $\eta \in S$, let α_η be the least $\alpha > \eta$ such that $L_\alpha[A \cap \eta] \models$ “$Z_3 + \eta = \omega_1$”.

We revise Harrington's shooting a club forcing notion as follows:

$$P_S^* = \{p \mid p \text{ is a closed bounded subset of } \omega_1, p \subseteq S \text{ and for all } \gamma \in p, \text{ if}$$
$$\text{there is } \alpha > \gamma \text{ such that } L_\alpha[A \cap \gamma, p \cap \gamma] \models \text{“}Z_3 + \gamma = \omega_1\text{”, then}$$
$$L_{\alpha_\gamma}[A \cap \gamma, p \cap \gamma] \models Z_3\}.$$

For $p, q \in P_S^*$, define $p \leq q \Leftrightarrow p$ end extends $q(i.e. p \supseteq q$ and for any $\alpha \in p \setminus q, \alpha > sup(q))$. Note that $P_S^* \in L_{\omega_1}[A]$ and $|P_S^*| = \omega_1$. So P_S^* is ω_2-c.c.

Lemma 2.3.42 *If $p \in P_S^*$, for any $\alpha < \omega_1$, there exists $q \in P_S^*$ such that $q \leq p$ and $\sup(q) > \alpha$.*

Proof Choose $\beta > \alpha$ such that $\beta > \sup(p), \beta \in S$ and $p \in L_\beta[A]$. Let $q = p \cup \{\beta\}$. Note that $p \in P_S^*, \beta > \sup(p)$ and $\beta \in S$. So q is a closed bounded subset of ω_1 and $q \subseteq S$. It suffices to show that $q \in P_S^*$. Since $p \in L_\beta[A], L_{\alpha_\beta}[A \cap \beta, q \cap \beta] = L_{\alpha_\beta}[A \cap \beta]$. So $L_{\alpha_\beta}[A \cap \beta, q \cap \beta] \models Z_3$ and $q \in P_S^*$. $\square$

Fact 2.3.43 *(Folklore, [18]) Suppose $M \models Z_3, P \in M$ is a forcing notion with $|P| \leq \omega_1$ and G is P-generic over M. If ω_1 is preserved, then $M[G] \models Z_3$.*

Fact 2.3.44 *(Folklore, [18]) Suppose P is a forcing notion and $|P| = \omega_1$. Then P is ω_1-distributive if and only if P adds no new reals.*[19]

[19] i.e. If G^* is P-generic over V, then $V[G], V$ have the same reals.

Lemma 2.3.45 *P_S^* is ω_1-distributive and so P_S^* preserves ω_1.*

Proof It suffices to show that P_S^* adds no new reals. Suppose G^* is P_S^*-generic over V. Let τ be the name for a new real in $V[G^*]$. Fix $p_0 \in P_S^*$. It suffices to show that there is $q \leq p_0$ such that $q \Vdash \tau \in V$. Let

$$R_\tau = \{(p, i, 1) \mid p \in P_S^* \wedge p \Vdash i \in \tau\} \cup \{(p, i, 0) \mid p \in P_S^* \wedge p \Vdash i \notin \tau\}.$$

From the definition of P_S^* and R_τ, $R_\tau \in H_{\omega_2}$. Take X such that

(a) $X \prec H_{\omega_2}, |X| = \omega$;

(b) $\{p_0, S, R_\tau, A\} \subseteq X$;

(c) $X \cap \omega_1 \in S$.

Since S is stationary, such X exists. Let $\gamma = X \cap \omega_1 \in S$. Note that $L_{\omega_2}[A] \models Z_3$. Since $H_{\omega_2} \models Z_3, X \models Z_3$ and so $X \cap L_{\omega_2}[A] \models Z_3$. The transitive collapse of $X \cap L_{\omega_2}[A]$ is in the form $L_\beta[A \cap \gamma]$ where $\gamma = X \cap \omega_1$ and $\beta = o.t.(X \cap \omega_2)$. So $L_\beta[A \cap \gamma] \models Z_3$. Let δ_γ be the least $\eta > \gamma$ such that $L_\eta[A \cap \gamma] \models Z_3$. Then $\delta_\gamma \leq \beta$. Note that α_γ is the least $\alpha > \gamma$ such that $L_\alpha[A \cap \gamma] \models$ "$Z_3 + \gamma = \omega_1$". So $\delta_\gamma \leq \alpha_\gamma$. Since $L_{\alpha_\gamma}[A \cap \gamma] \models \gamma = \omega_1$ and $\delta_\gamma \leq \alpha_\gamma$, we have $L_{\delta_\gamma}[A \cap \gamma] \models \gamma = \omega_1$. Since $L_{\delta_\gamma}[A \cap \gamma] \models$ "$Z_3 + \gamma = \omega_1$", we have $\alpha_\gamma \leq \delta_\gamma$ and hence $\alpha_\gamma = \delta_\gamma$. Since $\alpha_\gamma \leq \beta$ and β is countable, α_γ is countable.

Since $P_S^* \in L_{\omega_1}[A], A \in X$ and $\gamma = \omega_1^{L_{\alpha_\gamma}[A\cap\gamma]}$, we have $X \cap P_S^* = P_{S\cap\gamma}^* \cap L_\gamma[A] = P_{S\cap\gamma}^* \cap L_\gamma[A\cap\gamma] = P_{S\cap\gamma}^* \cap L_{\alpha_\gamma}[A\cap\gamma] = (P_{S\cap\gamma}^*)^{L_{\alpha_\gamma}[A\cap\gamma]}$. So $X \cap P_S^* \in L_{\alpha_\gamma}[A\cap\gamma]$. Take $g \subseteq X \cap P_S^*$ such that $p_0 \in g$ and g is $(P_{S\cap\gamma}^*)^{L_{\alpha_\gamma}[A\cap\gamma]}$-generic over $L_{\alpha_\gamma}[A\cap\gamma]$. Since $L_{\alpha_\gamma}[A\cap\gamma]$ is countable, such g exists in V. Let $q = \bigcup\{p \mid p \in g\} \cup \{\gamma\}$.

Claim

$$q \in P_S^*.$$

Proof Since $g \subseteq X \cap P_S^*, X$ is countable, g is $(P_{S\cap\gamma}^*)^{L_{\alpha_\gamma}[A\cap\gamma]}$-generic and $\gamma = X \cap \omega_1 \in S$, by the definition of P_S^* and q, we have q is closed bounded, $q \subseteq S$ and $\sup(q) = \gamma$. Fix $\gamma_0 \in q$. We show that if there is $\alpha > \gamma_0$ such that $L_\alpha[A\cap\gamma_0, q\cap\gamma_0] \models$ "$Z_3 + \gamma_0 = \omega_1$", then $L_{\alpha_{\gamma_0}}[A\cap\gamma_0, q\cap\gamma_0] \models Z_3$. If $\gamma_0 < \gamma = \sup(q)$, then $\gamma_0 \in p$ for some $p \in g$. It suffices to check the case that $\gamma_0 = \gamma$. Suppose there exists $\alpha > \gamma$ such that $L_\alpha[A\cap\gamma, q\cap\gamma] \models$ "$Z_3 + \gamma = \omega_1$"(if no such α exists, then we are done). Since $L_\alpha[A\cap\gamma] \models$ "$Z_3 + \gamma = \omega_1$", we have $\alpha \geq \alpha_\gamma$. So $L_{\alpha_\gamma}[A\cap\gamma, q\cap\gamma] \models \gamma = \omega_1$. Note that $L_{\alpha_\gamma}[A\cap\gamma, q\cap\gamma] = L_{\alpha_\gamma}[A\cap\gamma][g]$ and g is $(P_{S\cap\gamma}^*)^{L_{\alpha_\gamma}[A\cap\gamma]}$-generic over $L_{\alpha_\gamma}[A\cap\gamma]$. Since $L_{\alpha_\gamma}[A\cap\gamma] \models$ "$Z_3+\gamma = \omega_1$" and $L_{\alpha_\gamma}[A\cap\gamma, q\cap\gamma] \models \gamma = \omega_1$, ω_1 is preserved. By Fact 2.3.43 we have $L_{\alpha_\gamma}[A\cap\gamma, q\cap\gamma] \models Z_3$. So $q \in P_S^*$.

□

Since $q \in P_S^*$, $q \Vdash \tau = \{i \mid \exists p \in g(p \Vdash i \in \tau)\}$ and $g \in V$, we have

$q \Vdash \tau \in V$. $\square$

Hence P_S^* preserves all cardinals. Suppose G^* is P_S^*-generic over V. Let $C = \bigcup\{p \mid p \in G^*\}$. Then $C \subseteq S$ is a club on ω_1. Let $D = \{\alpha \mid \alpha$ is a limit point of $C\}$. From the definition of P_S^*, it is easy to see that P_S^* has the following property:

$$\text{For any } \eta \in D, \text{ if there is } \alpha > \eta \text{ such that}$$
$$L_\alpha[A \cap \eta, C \cap \eta] \models \text{“}Z_3 + \eta = \omega_1\text{”, then } L_{\alpha_\eta}[A \cap \eta, C \cap \eta] \models Z_3.$$

We want the club $C \subseteq S$ we add to have the following property:

$$\text{For any } \eta \in D, \text{ if } L_{\alpha_\eta}[A \cap \eta, C \cap \eta] \models \eta = \omega_1, \text{ then } L_{\alpha_\eta}[A \cap \eta, C \cap \eta] \models Z_3.$$

We did not see that P_S^* gives us such property. Note that for $\eta \in D$, if $L_{\alpha_\eta}[A \cap \eta, C \cap \eta] \models \eta = \omega_1$ implies $L_{\alpha_\eta}[A \cap \eta, C \cap \eta] \models Z_3$, then $\exists \alpha > \eta(L_\alpha[A \cap \eta, C \cap \eta] \models \text{“}Z_3 + \eta = \omega_1\text{”})$ implies $L_{\alpha_\eta}[A \cap \eta, C \cap \eta] \models Z_3$ since “$\alpha > \eta \wedge L_\alpha[A \cap \eta, C \cap \eta] \models Z_3 + \eta = \omega_1$” implies $\alpha \geq \alpha_\eta$ and so $L_{\alpha_\eta}[A \cap \eta, C \cap \eta] \models \eta = \omega_1$. But the converse is not true.

So we need to modify our forcing notion. In the following we use Baumgartner's forcing P_S^B defined in Section 2.3.2 and show that P_S^B gives us the desired property. Note that by Lemma 2.3.28, P_S^B is proper since S contains a club on ω_1.

Suppose G^* is P_S^B-generic over V. Define $F_{G^*} = \bigcup\{f \mid f \in G^*\}$. Then $F_{G^*} : \omega_1 \to S$ is increasing and continuous. Let $C = ran(F_{G^*})$. Then $C \subseteq S$ is a club on ω_1. In the rest part of this Chapter, $C \subseteq S$ always denotes the club forced via P_S^B and $D = \{\alpha \mid \alpha$ is a limit point of $C\}$. Note that $D \subseteq C$ is a club on ω_1.

Theorem 2.3.46 *Suppose $\eta \in S$ and $\{(\eta,\eta)\} \in P_S^B$. Then*

$$(P^B_{S\cap\eta})^{L_{\alpha_\eta}[A\cap\eta]} = \{f \in P_S^B \mid dom(f) \subseteq \eta \wedge ran(f) \subseteq \eta\}.$$

Proof The $\subseteq$ direction is trivial and we only show the $\supseteq$ direction.

($\supseteq$) Fix $g \in P_S^B$ with $dom(g) \subseteq \eta$ and $ran(g) \subseteq \eta$. We have to show that $g \in (P^B_{S\cap\eta})^{L_{\alpha_\eta}[A\cap\eta]}$. Let $\xi = \max(dom(g))$. Since $dom(g) \subseteq \eta, \xi < \eta$. Let $H : \xi + 1 \to S$ be a witness function for $g \in P_S^B$. Note that $H(\xi) = g(\xi) < \eta$ and for any $\alpha < \xi, H(\alpha) \in S \cap \eta$. It suffices to find an increasing continuous $\pi : \xi + 1 \to S \cap \eta$ such that $\pi \restriction dom(g) = g$ and $\pi \in L_{\alpha_\eta}[A \cap \eta]$.

Fix a surjection $e_0 : \omega \to \xi + 1$ such that $e_0 \in L_{\alpha_\eta}[A \cap \eta]$ and for any $\alpha \leq \xi, \{i \in \omega \mid e_0(i) = \alpha\}$ is infinite. Fix a surjection $e_1 : \omega \to H(\xi) + 1$ such that $e_1 \in L_{\alpha_\eta}[A \cap \eta]$. Let T be the set of all pairs (π_1, π_2) where $\pi_1 : k \to (H(\xi) + 1) \cap S$ and $\pi_2 : k \to \omega$ for some $k \in \omega$ with the following properties:

(1) For all $i < k$, if $e_0(i) \in dom(g)$, then $\pi_1(i) = g(e_0(i))$.

(2) For all $i < j < k$, $\pi_1(i) = \pi_1(j)$ if and only if $e_0(i) = e_0(j)$.

(3) For all $i < j < k$, $\pi_1(i) < \pi_1(j)$ if and only if $e_0(i) < e_0(j)$.

(4) For all $i < k$, if $e_0(i) > 0$ and $e_0(i)$ is a limit ordinal, then $\pi_1(i) > 0$ and $\pi_1(i)$ is a limit ordinal.

(5) For all $i < k$, if $e_0(i) > 0$ is a limit ordinal and $\pi_2(i) < k$, then

$$\sup(\{e_1(m) \mid m \leq i \wedge e_1(m) < \pi_1(i)\}) < \pi_1(\pi_2(i)) < \pi_1(i)$$

and $e_0(\pi_2(i)) < e_0(i)$.

Define the order on T as follows: if (π_1', π_2'), $(\pi_1, \pi_2) \in T$, then $(\pi_1', \pi_2') \leq (\pi_1, \pi_2)$ if and only if $dom(\pi_1) \subseteq dom(\pi_1'), dom(\pi_2) \subseteq dom(\pi_2'), \pi_1' \restriction dom(\pi_1) = \pi_1$ and $\pi_2' \restriction dom(\pi_2) = \pi_2$. Since $S \cap \eta, e_0, e_1, g \in L_{\alpha_\eta}[A \cap \eta]$, from the definition of T, $T \in L_{\alpha_\eta}[A \cap \eta]$.

Lemma 2.3.47 *Suppose $\langle(\pi_1^n, \pi_2^n) \mid n \in \omega\rangle$ is a descending sequence from T. Then there is an increasing continuous $H^\infty : \xi + 1 \to S \cap \eta$ such that $H^\infty \restriction dom(g) = g$ and $H^\infty \in L_{\alpha_\eta}[A \cap \eta]$.*

Proof Let $\pi_1^\infty = \bigcup_{n\in\omega} \pi_1^n$ and $\pi_2^\infty = \bigcup_{n\in\omega} \pi_2^n$. From the definition of T, $\pi_1^\infty : \omega \to (H(\xi) + 1) \cap S$ and $\pi_2^\infty : \omega \to \omega$. Define $H^\infty : \xi + 1 \to S \cap \eta$ by $H^\infty(e_0(i)) = \pi_1^\infty(i)$ for any $i \in \omega$. We show that H^∞ is the function we want. By (2), H^∞ is well defined. By (3), H^∞ is increasing. By (1), H^∞ extends g.

Since $T, e_0 \in L_{\alpha_\eta}[A \cap \eta]$, by the definition of H^∞, $H^\infty \in L_{\alpha_\eta}[A \cap \eta]$. It suffices to show that H^∞ is continuous. Since for any $i \in \omega$, we can find large enough $n \in \omega$ such that $i < dom(\pi_1^n) = dom(\pi_2^n)$ and $\pi_2^n(i) < dom(\pi_1^n) = dom(\pi_2^n)$, by (5), we have:

$$\text{For all } i < \omega, \text{ if } e_0(i) > 0 \text{ is a limit ordinal, then}$$
$$\sup(\{e_1(m) \mid m \leq i \wedge e_1(m) < \pi_1^\infty(i)\}) < \pi_1^\infty(\pi_2^\infty(i)) < \pi_1^\infty(i) \text{ and}$$
$$e_0(\pi_2^\infty(i)) < e_0(i).$$

Claim H^∞ is continuous.

Proof Suppose $0 < \alpha \leq \xi$ is a limit ordinal. We show that $H^\infty(\alpha) = \sup(\{H^\infty(\beta) \mid \beta < \alpha\})$. Suppose not. Then there exists θ such that $\sup(\{H^\infty(\beta) \mid \beta < \alpha\}) < \theta < H^\infty(\alpha)$.

Fix m_0 such that $e_1(m_0) = \theta$. Such m_0 exists since e_1 is surjective. Fix $i > m_0$ such that $e_0(i) = \alpha$. Such i exists since $\{i \in \omega \mid e_0(i) = \alpha\}$ is infinite. Note that

$$\theta \leq \sup(\{e_1(m) \mid m \leq i \wedge e_1(m) < \pi_1^\infty(i)\}) < \pi_1^\infty(\pi_2^\infty(i)) < \pi_1^\infty(i) = H^\infty(\alpha)$$

since $\theta = e_1(m_0) < H^\infty(\alpha) = \pi_1^\infty(i)$ and $e_0(i)$ is limit. Note that

$$\pi_1^\infty(\pi_2^\infty(i)) = H^\infty(e_0(\pi_2^\infty(i))) < H^\infty(e_0(i)) = H^\infty(\alpha)$$

since $e_0(\pi_2^\infty(i)) < e_0(i)$. So $\theta < H^\infty(e_0(\pi_2^\infty(i)))$ where $e_0(\pi_2^\infty(i)) < \alpha$. But $\sup(\{H^\infty(\beta) \mid \beta < \alpha\}) < \theta$. Contradiction. $\square$

$\square$

Lemma 2.3.48 *There exists a descending sequence $\langle(\pi_1^n, \pi_2^n) \mid n \in \omega\rangle$ from T such that if $\pi_1^\infty = \bigcup_{n\in\omega} \pi_1^n, \pi_2^\infty = \bigcup_{n\in\omega} \pi_2^n$ and $H^\infty(e_0(i)) = \pi_1^\infty(i)$ for any $i \in \omega$, then $H^\infty = H$.*

Proof Define $\pi_1^\infty(i) = H(e_0(i))$ for any $i \in \omega$. Now we define π_2^∞ as follows such that for all $i < \omega$, if $e_0(i) > 0$ is a limit ordinal, then $\sup(\{e_1(m) \mid m \leq i \wedge e_1(m) < \pi_1^\infty(i)\}) < \pi_1^\infty(\pi_2^\infty(i)) < \pi_1^\infty(i)$ and $e_0(\pi_2^\infty(i)) < e_0(i)$. Fix $i < \omega$ such that $e_0(i) > 0$ is a limit ordinal. Let $\alpha = e_0(i)$. Note that $\pi_1^\infty(i) = H(e_0(i)) = H(\alpha)$. Since H is continuous, $H(\alpha)$ is a limit ordinal. So we can take $\beta < \alpha$ such that $\sup(\{e_1(m) \mid m \leq i \wedge e_1(m) < \pi_1^\infty(i)\}) < H(\beta) < H(\alpha)$. Let $\pi_2^\infty(i) =$ the least $j \in \omega$ such that $e_0(j) = \beta$. Note that $\sup(\{e_1(m) \mid m \leq i \wedge e_1(m) < \pi_1^\infty(i)\}) < \pi_1^\infty(\pi_2^\infty(i)) < \pi_1^\infty(i)$ and $e_0(\pi_2^\infty(i)) < e_0(i)$ since $\pi_1^\infty(\pi_2^\infty(i)) = \pi_1^\infty(j) = H(e_0(j)) = H(\beta), \pi_1^\infty(i) = H(\alpha)$ and $e_0(\pi_2^\infty(i)) = \beta < \alpha$ where $\alpha = e_0(i)$.

Claim For any $k \in \omega, (\pi_1^\infty \restriction k, \pi_2^\infty \restriction k) \in T$.

Proof We need to check that for any $k \in \omega, (\pi_1^\infty \restriction k, \pi_2^\infty \restriction k)$ satisfies conditions $(1) - (5)$ in the definition of T. Since H extends $g,(1)$ holds. Since H is strictly increasing, (2) and (3) holds. Since H is continuous, (4) holds. From our definition of π_1^∞ and π_2^∞, (5) holds. $\square$

From the definition of H^∞ and π_1^∞, since e_0 is surjective, if $H^\infty(e_0(i)) = \pi_1^\infty(i)$ for any $i \in \omega$, then $H^\infty = H$. $\square$

Let $\langle(\pi_1^n, \pi_2^n) \mid n \in \omega\rangle$ be a descending sequence from T. By Lemma 2.3.47, $\langle(\pi_1^n, \pi_2^n) \mid n \in \omega\rangle$ induces a witness function for $g \in (P^B_{S\cap\eta})^{L_{\alpha_\eta}[A\cap\eta]}$.

$\square$

Theorem 2.3.49 *Suppose $\eta \in S$ and $f \in P^B_S$ where $f = \{(\eta, \eta)\}$. Then*

$$(P^B_S)_f = \{g \cup \{(\eta, \eta)\} \mid g \in (P^B_{S\cap\eta})^{L_{\alpha_\eta}[A\cap\eta]}\}.$$

Proof Follows from Theorem 2.3.46 and Lemma 2.3.24 since for any $\eta \in S, \eta$ is indecomposable. $\square$

Theorem 2.3.50 *For any $\eta \in D$, $L_{\alpha_\eta}[A \cap \eta, C \cap \eta] \models \eta = \omega_1$ if and only if $L_{\alpha_\eta}[A \cap \eta, C \cap \eta] \models Z_3$.*

Proof ($\Rightarrow$) Fix $\eta \in D$. Since $L_{\alpha_\eta}[A\cap\eta, C\cap\eta] \models \eta = \omega_1$, $L_{\alpha_\eta}[A\cap\eta, C\cap\eta] \models$ "$C \cap \eta$ is a club on η" and hence $o.t.(C \cap \eta) = \eta$. So η is the η-th element of C. Since $F_{G^*}(\xi)$ is the ξ-th element of C, $F_{G^*}(\eta) = \eta$. Let $Q = \{f \in P^B_S \mid \eta = \max(dom(f)) \wedge f(\eta) = \eta\}$. Note that $Q = (P^B_S)_f$ where $dom(f) = \{\eta\}$ and $f(\eta) = \eta$. Since $f = \{(\eta, \eta)\} \in G^*, G^* \cap Q$ is Q-generic over V. Note that $Q = \{h \cup \{\eta, \eta\} \mid h \in (P^B_{S\cap\eta})^{L_{\alpha_\eta}[A\cap\eta]}\}$ by Theorem 2.3.49.

Suppose $L_{\alpha_\eta}[A\cap\eta] \models S\cap\eta$ is not stationary on η. Then there exists a club $E\subseteq\eta$ on η such that $E\in L_{\alpha_\eta}[A\cap\eta]$ and $E\cap S\cap\eta=\emptyset$. So $E\cap C\cap\eta=\emptyset$. Since $L_{\alpha_\eta}[A\cap\eta, C\cap\eta]\models$ "$\eta=\omega_1$ and $E, C\cap\eta$ are disjoint closed subsets of η". Contradiction.

So $L_{\alpha_\eta}[A\cap\eta]\models S\cap\eta$ is stationary on η. Since $G^*\cap Q$ is Q-generic over V and $Q=\{h\cup\{\eta,\eta\}\mid h\in (P^B_{S\cap\eta})^{L_{\alpha_\eta}[A\cap\eta]}\}$, we have $G^*\cap(P^B_{S\cap\eta})^{L_{\alpha_\eta}[A\cap\eta]}$ is $(P^B_{S\cap\eta})^{L_{\alpha_\eta}[A\cap\eta]}$-generic over $L_{\alpha_\eta}[A\cap\eta]$. Do Baumgartner's forcing over $P^B_{S\cap\eta}$ in $L_{\alpha_\eta}[A\cap\eta]$. Note that $L_{\alpha_\eta}[A\cap\eta, C\cap\eta]=L_{\alpha_\eta}[A\cap\eta, G^*\cap(P^B_{S\cap\eta})^{L_{\alpha_\eta}[A\cap\eta]}]$ and $L_{\alpha_\eta}[A\cap\eta]\models$ "$|(P^B_{S\cap\eta})^{L_{\alpha_\eta}[A\cap\eta]}|=\omega_1$ and $(P^B_{S\cap\eta})^{L_{\alpha_\eta}[A\cap\eta]}$ preserves ω_1". So $L_{\alpha_\eta}[A\cap\eta, C\cap\eta]\models Z_3$ since $G^*\cap(P^B_{S\cap\eta})^{L_{\alpha_\eta}[A\cap\eta]}$ is $(P^B_{S\cap\eta})^{L_{\alpha_\eta}[A\cap\eta]}$-generic over $L_{\alpha_\eta}[A\cap\eta]$.

($\Leftarrow$) We show that if $L_{\alpha_\eta}[A\cap\eta, C\cap\eta]\models Z_3$, then $L_{\alpha_\eta}[A\cap\eta, C\cap\eta]\models\eta=\omega_1$. Suppose not. i.e. $\eta<\omega_1^{L_{\alpha_\eta}[A\cap\eta, C\cap\eta]}$. Since $L_{\alpha_\eta}[A\cap\eta]\subseteq L_{\alpha_\eta}[A\cap\eta, C\cap\eta]$, $\omega_1^{L_{\alpha_\eta}[A\cap\eta, C\cap\eta]}$ is a cardinal in $L_{\alpha_\eta}[A\cap\eta]$. But since $L_{\alpha_\eta}[A\cap\eta]\models$ "$Z_3+\eta=\omega_1$", $\eta=\omega_1^{L_{\alpha_\eta}[A\cap\eta]}$ is the largest cardinal in $L_{\alpha_\eta}[A\cap\eta]$. Contradiction. $\square$

In fact, we have proved that for any $\eta\in S$, if $L_{\alpha_\eta}[A\cap\eta]\models S\cap\eta$ is stationary on η, then for any $f\in P^B_S$ such that $\eta\in dom(f)$ and $f(\eta)=\eta$, we have $f\Vdash L_{\alpha_\eta}[A\cap\eta, C\cap\eta]\models Z_3$.

Corollary 2.3.51 *For any $\eta \in D$, the following are equivalent:*

(i) $L_{\alpha_\eta}[A \cap \eta, C \cap \eta] \models \eta = \omega_1$.

(ii) $L_{\alpha_\eta}[A \cap \eta] \models S \cap \eta$ *is stationary.*

(iii) $C \cap \eta$ *is* $(P^B_{S\cap\eta})^{L_{\alpha_\eta}[A\cap\eta]}$*-generic over* $L_{\alpha_\eta}[A \cap \eta]$.

(iv) $L_{\alpha_\eta}[A \cap \eta, C \cap \eta] \models Z_3$.

As a summary, D has the following properties:

- For any $\eta \in D, \eta$ is an L-cardinal.
- For any $\eta \in D$, if $\eta \leq \beta < \alpha_\eta$ and β is $A \cap \eta$-admissible, then β is an L-cardinal.
- For any $\eta \in D$, $L_{\alpha_\eta}[A \cap \eta, C \cap \eta] \models \eta = \omega_1$ if and only if $L_{\alpha_\eta}[A \cap \eta, C \cap \eta] \models Z_3$.
- For all $\eta \in D$, if there is $\alpha > \eta$ such that $L_\alpha[A \cap \eta, C \cap \eta] \models$ "$Z_3 + \eta = \omega_1$", then $L_{\alpha_\eta}[A \cap \eta, C \cap \eta] \models Z_3$.

2.3.8 Step Five

Now we work in $L[G][H][A][C]$. Since $L[A, C] \models ZFC$, there exists $\alpha > \omega_1$ such that $L_\alpha[A, C] \models Z_3$. Take $X \prec L_\alpha[A, C]$ such that $|X| = \omega$ and $X \cap \omega_1 \in D$. Since D is a club on ω_1, such X exists. Let $\eta = X \cap \omega_1$.

The transitive collapse of X is in the form $L_{\bar{\alpha}}[A \cap \eta, C \cap \eta]$. Note that $\omega_1^{L_{\bar{\alpha}}[A\cap\eta,C\cap\eta]} = \eta$. So $L_{\bar{\alpha}}[A \cap \eta, C \cap \eta] \models$ "$Z_3 + \eta = \omega_1$". Since $\eta \in D$, by the property of D, $L_{\alpha_\eta}[A\cap\eta, C\cap\eta] \models$ "$Z_3+\eta = \omega_1$". So we have shown that there exists $\eta \in D$ such that $L_{\alpha_\eta}[A\cap\eta, C\cap\eta] \models$ "$Z_3+\eta = \omega_1$". Let η^* be the least $\eta \in D$ such that $L_{\alpha_\eta}[A\cap\eta, C\cap\eta] \models$ "$Z_3+\eta = \omega_1$". So $L_{\alpha_{\eta^*}}[A\cap\eta^*, C\cap\eta^*] \models Z_3$ and $\eta^* = \omega_1^{L_{\alpha_{\eta^*}}[A\cap\eta^*,C\cap\eta^*]}$. Note that η^* is a limit point of D. (Suppose not. Let ξ be the largest $\xi \in D$ such that $\xi < \eta^*$. Then $o.t.(C\cap(\eta^* \setminus (\xi+1))) = \omega$. But since $L_{\alpha_{\eta^*}}[A\cap\eta^*, C\cap\eta^*] \models \eta^* = \omega_1$, $L_{\alpha_{\eta^*}}[A\cap\eta^*, C\cap\eta^*] \models C\cap\eta^*$ is a club on η^*. Contradiction.)

Lemma 2.3.52 *(Basic properties of D)*

(1) If $\eta \in D$ and $\eta < \eta^*$, then $L_{\alpha_\eta}[A\cap\eta, C\cap\eta] \models \eta < \omega_1$.

(2) Suppose $\eta \in D, \eta < \eta^*$ and $\beta < \alpha_\eta$. Then $L_{\alpha_\eta}[A\cap\eta, C\cap\eta] \models \beta < \omega_1$.

(3) Suppose $\eta \in S$ and $\beta < \eta$. Then $L_\eta[A] \models \beta < \omega_1$.

(4) If $\eta_0, \eta_1 \in S$ and $\eta_0 < \eta_1$, then $\alpha_{\eta_0} < \eta_1$. i.e. For any $\eta \in S, \alpha_\eta < \bar{\eta}$ where $\bar{\eta} = \min(S \setminus (\eta+1))$.

Proof (1) Suppose $L_{\alpha_\eta}[A\cap\eta, C\cap\eta] \models \eta = \omega_1$. By the property of D, $L_{\alpha_\eta}[A\cap\eta, C\cap\eta] \models Z_3$. Since η^* is the least $\eta \in D$ such that $L_{\alpha_\eta}[A\cap\eta, C\cap\eta] \models$ "$Z_3 + \eta = \omega_1$". So $\eta \geq \eta^*$. Contradiction.

(2) Since $L_{\alpha_\eta}[A\cap\eta] \models Z_3, L_{\alpha_\eta}[A\cap\eta] \models \omega_2$ does not exist. So $L_{\alpha_\eta}[A\cap\eta] \models \forall\beta \in Ord(|\beta| \leq \omega_1)$. Since $L_{\alpha_\eta}[A\cap\eta] \models \eta = \omega_1$ and $\beta < \alpha_\eta$, there exists $f \in L_{\alpha_\eta}[A\cap\eta]$ such that $f : \eta \to \beta$ is surjective. Since $L_{\alpha_\eta}[A\cap\eta, C\cap\eta] \models \eta < \omega_1$, there exists $g \in L_{\alpha_\eta}[A\cap\eta, C\cap\eta]$ such that $g : \omega \to \eta$ is surjective. So $f \circ g : \omega \to \beta$ is surjective and $f \circ g \in L_{\alpha_\eta}[A\cap\eta, C\cap\eta]$. So $L_{\alpha_\eta}[A\cap\eta, C\cap\eta] \models \beta < \omega_1$.

(3) Since $\eta \in S, L_{\alpha_\eta}[A\cap\eta] \models \eta = \omega_1$. Note that $\mathbb{R} \cap L_{\alpha_\eta}[A\cap\eta] = \mathbb{R} \cap L_\eta[A\cap\eta] = \mathbb{R} \cap L_\eta[A]$. Since $\beta < \eta, L_\eta[A] = L_\eta[A\cap\eta] \models \beta < \omega_1$.

(4) Suppose $\eta_1 \leq \alpha_{\eta_0}$. Note that $Z_3 \vdash \forall E \subseteq \omega_1(L_{\omega_1}[E] \models ZFC^-)$. Since $L_{\alpha_{\eta_1}}[A\cap\eta_1] \models$ "$Z_3 + \eta_1 = \omega_1$", $L_{\eta_1}[A\cap\eta_0] \models ZFC^-$. Since $\eta_1 \leq \alpha_{\eta_0}, L_{\eta_1}[A\cap\eta_0] \subseteq L_{\alpha_{\eta_0}}[A\cap\eta_0]$. Since $L_{\alpha_{\eta_0}}[A\cap\eta_0] \models Z_3, L_{\eta_1}[A\cap\eta_0] \models \eta_0 = \omega_1$ and $L_{\eta_1}[A\cap\eta_0] \models ZFC^-$, we have $L_{\eta_1}[A\cap\eta_0] \models Z_3$. So $\eta_1 \geq \alpha_{\eta_0}$ and hence $\eta_1 = \alpha_{\eta_0}$.

Fact 2.3.53 *(Folklore, [9], [12])* (Z_3) $\forall E \subseteq \omega_1 \forall\alpha < \omega_1 \forall a \in L_{\omega_1}[E] \exists X(X \prec L_{\omega_1}[E] \wedge |X| = \omega \wedge \alpha \cup \{a\} \subseteq X)$.

Since $L_{\alpha_{\eta_1}}[A\cap\eta_1] \models Z_3$, applying the fact with $E = A\cap\eta_0, \alpha = \eta_0$ and $a = A\cap\eta_0$, there is $X \in L_{\alpha_{\eta_1}}[A\cap\eta_1]$ such that $X \prec L_{\eta_1}[A\cap\eta_0], L_{\alpha_{\eta_1}}[A\cap\eta_1] \models |X| = \omega, A\cap\eta_0 \in X$ and $\eta_0 \subseteq X$. Let M be the transitive collapse of X. Then

$M = L_{\bar{\eta_1}}[A\cap\eta_0]$ where $\bar{\eta_1}$ is the image of $X\cap\eta_1$ under the transitive collapse of X. Since $L_{\alpha_{\eta_1}}[A\cap\eta_1] \models |X| = \omega$ and $L_{\alpha_{\eta_1}}[A\cap\eta_1] \models \eta_1 = \omega_1$, we have $\bar{\eta_1} < \eta_1$. Note that $L_{\eta_1}[A\cap\eta_0] \models Z_3+\eta_0 = \omega_1$. So $L_{\bar{\eta_1}}[A\cap\eta_0] \models Z_3+\eta_0 = \omega_1$. Hence $\alpha_{\eta_0} \leq \bar{\eta_1} < \eta_1$. Contradiction. $\square$

Now we work in $L_{\alpha_{\eta^*}}[A\cap\eta^*, C\cap\eta^*]$. We first define an almost disjoint system $\langle\delta_\beta : \beta < \eta^*\rangle$ on ω and $B \subseteq \eta^*$ in $L_{\alpha_{\eta^*}}[A\cap\eta^*, C\cap\eta^*]$ and then do almost disjoint forcing in $L_{\alpha_{\eta^*}}[A\cap\eta^*, C\cap\eta^*]$ to build a generic real x to code B relative to $\langle\delta_\beta : \beta < \eta^*\rangle$.

To define an almost disjoint system $\langle\delta_\beta : \beta < \eta^*\rangle$ we first define $\langle f_\beta : \beta < \eta^*\rangle$ in $L_{\alpha_{\eta^*}}[A\cap\eta^*, C\cap\eta^*]$ by induction on $\beta < \eta^*$. Fix $\beta < \eta^*$. We define f_β as follows.

- Suppose $\beta < \omega$. Let $f_\beta : \omega \to 1+\beta$ be a recursive function.
- Suppose $\beta \geq \omega$. Let $\eta_0 = \sup(D\cap\beta)$ and η_1 be the least ordinal in C such that $\eta_1 > \beta$. Note that $\eta_0 = 0$ or $\eta_0 \in D$. Suppose $\eta_0 = 0$. Since $\eta_1 \in C$ and $\beta < \eta_1, L_{\eta_1}[A] \models \beta < \omega_1$. Let $f_\beta : \omega \to \beta$ be the least surjection in $L_{\eta_1}[A]$.
- Suppose $\eta_0 \neq 0$ and $\beta < \alpha_{\eta_0}$. Since $\eta_0 \in D, \eta_0 < \eta^*$ and $\beta < \alpha_{\eta_0}, L_{\alpha_{\eta_0}}[A\cap\eta_0, C\cap\eta_0] \models \beta < \omega_1$. Let $f_\beta : \omega \to \beta$ be the least surjection in $L_{\alpha_{\eta_0}}[A\cap\eta_0, C\cap\eta_0]$.

- Suppose $\eta_0 \neq 0$ and $\beta \geq \alpha_{\eta_0}$. Note that $\alpha_{\eta_0} < \eta_1$. Since $\eta_1 \in S$ and $\beta < \eta_1, L_{\eta_1}[A] \models \beta < \omega_1$. Let $f_\beta : \omega \to \beta$ be the least surjection in $L_{\eta_1}[A]$.

Now we define an almost disjoint system $\langle \delta_\beta : \beta < \eta^* \rangle$ on ω in $L_{\alpha_{\eta^*}}[A \cap \eta^*, C \cap \eta^*]$ from $\langle f_\beta : \beta < \eta^* \rangle$ as follows. Fix a recursive bijection $\pi : \omega \leftrightarrow \omega \times \omega$. Let $x_\beta = \{(i,j) \mid f_\beta(i) < f_\beta(j)\}$. Let $y_\beta = \{k \in \omega \mid \pi(k) \in x_\beta\}$. Now we have $\langle y_\beta \mid \beta < \eta^* \rangle$. Note that for $\alpha, \beta < \eta^*$, if $\alpha \neq \beta$, then $y_\alpha \neq y_\beta$ (in fact $y_\alpha \nsubseteq y_\beta$ and $y_\beta \nsubseteq y_\alpha$). So $\langle y_\beta \mid \beta < \eta^* \rangle$ is a sequence of distinct reals. Let $\langle s_i \mid i \in \omega \rangle$ be an injective and recursive enumeration of $\omega^{<\omega}$. For any $\beta < \eta^*$, define

$$\delta_\beta = \{i \in \omega \mid \exists m \in \omega(s_i = y_\beta \cap m)\}.$$

From the definition, for any $\alpha, \beta \in \eta^*, \delta_\beta$ is infinite and if $\alpha \neq \beta$, then $\delta_\alpha \cap \delta_\beta$ is finite. So $\langle \delta_\beta : \beta < \eta^* \rangle$ is a sequence of almost disjoint reals. Since $\langle s_i \mid i \in \omega \rangle$ is recursive, π is recursive and for any $i \in \omega, f_i$ is recursive, we have $\langle \delta_i : i \in \omega \rangle$ is recursive.

Now we define $B \subseteq \eta^*$ in $L_{\alpha_{\eta^*}}[A \cap \eta^*, C \cap \eta^*]$ as follows. Fix $\beta < \eta^*$. We define z_β as follows. Let η_0^β be the least $\eta \in D$ such that $\eta > \beta$ and η_1^β be the least $\eta \in D$ such that $\eta > \eta_0^\beta$. Note that $\eta_1^\beta < \eta^*$ since $\beta < \eta^*$ and η^* is a limit point of D. So $\alpha_{\eta_0^\beta}$ is countable in $L_{\alpha_{\eta_1^\beta}}[A \cap \eta_1^\beta, C \cap \eta_1^\beta]$. Let z_β be the least real in $L_{\alpha_{\eta_1^\beta}}[A\cap\eta_1^\beta, C\cap\eta_1^\beta]$ which codes $\langle\eta_0^\beta, L_{\alpha_{\eta_0^\beta}}[A\cap\eta_0^\beta, C\cap\eta_0^\beta], A\cap\eta_0^\beta, C\cap\eta_0^\beta\rangle$. Such z_β exists. Now we get a sequence $\langle z_\beta \mid \beta<\eta^*\rangle$. Note that if $\beta_0<\beta_1<\eta^*$, then z_{β_0} is recursive in z_{β_1}. Define

$$B=\{\omega\cdot\alpha+i \mid \alpha<\eta^*\wedge i\in z_\alpha\}.$$

From the definition of $\langle\delta_\beta:\beta<\eta^*\rangle$ and $\langle z_\beta \mid \beta<\eta^*\rangle$, we have:

- If $\eta\in D$ and $\eta\le\eta^*$, then $\langle\delta_\beta:\beta<\eta\rangle$ is definable in $\langle L_\eta[A\cap\eta, C\cap\eta], A\cap\eta, C\cap\eta\rangle$.
- If η is a limit point of D, then $\langle z_\beta:\beta<\eta\rangle$ is definable in $\langle L_\eta[A\cap\eta, C\cap\eta], A\cap\eta, C\cap\eta\rangle$.

Note that if η is a limit point of D, then $\omega\cdot\beta<\eta$ for all $\beta<\eta$ and $B\cap\eta=\{\omega\cdot\beta+i \mid i\in z_\beta \text{ and } \beta<\eta\}$. So if η is a limit point of D, $B\cap\eta$ is definable in $\langle L_\eta[A\cap\eta, C\cap\eta], A\cap\eta, C\cap\eta\rangle$.

By a *c.c.c* almost disjoint forcing we can build a generic real $x\in 2^\omega$ over $L_{\alpha_{\eta^*}}[A\cap\eta^*, C\cap\eta^*]$ to code B relative to $\langle\delta_\beta:\beta<\eta^*\rangle$ such that for any $\alpha<\eta^*, \alpha\in B\Leftrightarrow |x\cap\delta_\alpha|<\omega$. Then $L_{\alpha_{\eta^*}}[A\cap\eta^*, C\cap\eta^*][x]\models Z_3$ since $L_{\alpha_{\eta^*}}[A\cap\eta^*, C\cap\eta^*]\models Z_3$ and x is a generic real built via a *c.c.c* forcing. Note that $L_{\alpha_{\eta^*}}[A\cap\eta^*, C\cap\eta^*][x]=L_{\alpha_{\eta^*}}[x]$ since x codes $(A\cap\eta^*, C\cap\eta^*)$ via $\langle\delta_\beta:\beta<\eta^*\rangle$.

In the following part of the argument, we assume that $\lambda < \alpha_{\eta^*}$ and λ is x-admissible. Since $\langle \delta_i \mid i \in \omega \rangle$ is recursive, $\langle \delta_i \mid i \in \omega \rangle \in L_\lambda[x]$. Note that $B \cap \omega = z_0$. Since $B \cap \omega = \{i \in \omega \mid |x \cap \delta_i| < \omega\}$, we have $z_0 \in L_\lambda[x]$. Since z_0 codes $\eta_0^0, \min(D) = \eta_0^0 < \lambda$.

Definition 2.3.54

$$\theta = \sup(\{\beta < \eta^* \mid z_\beta \in L_\lambda[x]\}); \quad \gamma = \sup(\{\eta_0^\beta \mid \beta < \theta\}).$$

Lemma 2.3.55 *θ is a limit ordinal.*

Proof We show that for $\beta < \eta^*$, if $\beta < \theta$, then $\beta + 1 < \theta$. It suffices to show that for $\beta < \eta^*$, if $z_\beta \in L_\lambda[x]$, then $z_{\beta+1} \in L_\lambda[x]$. Fix $\beta < \eta^*$. Suppose $z_\beta \in L_\lambda[x]$. Since z_β codes $\langle \eta_0^\beta, L_{\alpha_{\eta_0^\beta}}[A \cap \eta_0^\beta, C \cap \eta_0^\beta], A \cap \eta_0^\beta, C \cap \eta_0^\beta \rangle$, we have $\langle \eta_0^\beta, L_{\alpha_{\eta_0^\beta}}[A \cap \eta_0^\beta, C \cap \eta_0^\beta], A \cap \eta_0^\beta, C \cap \eta_0^\beta \rangle \in L_\lambda[x]$. Note that $\langle \delta_\xi \mid \eta_0^\beta \leq \xi < \alpha_{\eta_0^\beta} \rangle \in L_\lambda[x]$ where $\eta_0^\beta = \sup(D \cap \xi)$ since if $\eta_0^\beta = \sup(D \cap \xi)$ and $\eta_0^\beta \leq \xi < \alpha_{\eta_0^\beta}$, then $\delta_\xi \in L_{\alpha_{\eta_0^\beta}}[A \cap \eta_0^\beta, C \cap \eta_0^\beta]$. Since $\eta_0^\beta \in D, \omega \cdot \eta_0^\beta = \eta_0^\beta$. So $z_{\eta_0^\beta} = \{i \in \omega \mid \omega \cdot \eta_0^\beta + i \in B\} = \{i \in \omega \mid \eta_0^\beta + i \in B\} = \{i \in \omega \mid |x \cap \delta_{\eta_0^\beta + i}| < \omega\}$. Since $\eta_0^\beta = \sup(D \cap (\eta_0^\beta + i))$ and $\eta_0^\beta + i < \alpha_{\eta_0^\beta}, \langle \delta_{\eta_0^\beta + i} \mid i \in \omega \rangle \in L_\lambda[x]$. So $z_{\eta_0^\beta} \in L_\lambda[x]$. Since $\beta < \eta_0^\beta, z_{\beta+1}$ is recursive in $z_{\eta_0^\beta}$. So $z_{\beta+1} \in L_\lambda[x]$. $\square$

Lemma 2.3.56 *$\langle z_\beta \mid \beta < \theta \rangle$ is Σ_1-definable in $L_\lambda[x]$ from x.*

Proof Note that if $\beta < \theta$, then $z_\beta \in L_\lambda[x]$ and z_β codes $\langle \eta_0^\beta, L_{\alpha_{\eta_0^\beta}}[A \cap \eta_0^\beta, C \cap \eta_0^\beta], A \cap \eta_0^\beta, C \cap \eta_0^\beta \rangle$. Since $z_\beta \in L_\lambda[x]$ and λ is x-admissible, there

exists $\lambda_0 < \lambda$ such that $\beta < \lambda_0, \lambda_0$ is a limit ordinal and $\langle \eta_0^\beta, L_{\alpha_{\eta_0^\beta}}[A \cap \eta_0^\beta, C \cap \eta_0^\beta], A \cap \eta_0^\beta, C \cap \eta_0^\beta \rangle \in L_{\lambda_0}[x]$. Note that there exists a formula $\phi_0(\alpha, x)$ which uniformly defines the well ordering $<_{L_\alpha[x]}$ on $L_\alpha[x]$. We can find a formula $\varphi(\alpha, z, \beta, x, A, C)$ which says that $\langle \eta_0^\beta, L_{\alpha_{\eta_0^\beta}}[A \cap \eta_0^\beta, C \cap \eta_0^\beta], A \cap \eta_0^\beta, C \cap \eta_0^\beta \rangle$ is countable in $L_\alpha[x]$ and z is the $<_{L_\alpha[x]}$-least real which codes $\langle \eta_0^\beta, L_{\alpha_{\eta_0^\beta}}[A \cap \eta_0^\beta, C \cap \eta_0^\beta], A \cap \eta_0^\beta, C \cap \eta_0^\beta \rangle$.
By absoluteness for any $\beta < \theta, z = z_\beta$ if and only if $\exists \lambda_0 < \lambda(z \in L_{\lambda_0}[x] \wedge \beta < \lambda_0 \wedge \lambda_0$ is a limit ordinal $\wedge L_{\lambda_0}[x] \models \varphi[\lambda_0, z, \beta, x, A, C])$. $\square$

Note that we assume that $\lambda < \alpha_{\eta^*}$ and λ is x-admissible.

Theorem 2.3.57 *λ is an L-cardinal.*

Proof If $\beta < \theta$, then $z_\beta \in L_\lambda[x]$ and $\beta < \eta_0^\beta < \lambda$ since z_β codes η_0^β. So $\theta \leq \lambda$.

Case 1: $\theta = \lambda$. Since $\beta < \eta_0^\beta < \lambda$ for any $\beta < \theta$, we have $\gamma = \sup(\{\eta_0^\beta \mid \beta < \theta\}) = \sup(\{\eta_0^\beta \mid \beta < \lambda\}) = \lambda$. Since $\gamma \in D, \lambda \in D$ and so λ is an L-cardinal.

Case 2: $\theta < \lambda$. Since $\langle z_\beta \mid \beta < \theta \rangle$ is Σ_1-definable in $L_\lambda[x]$ from x and $\theta < \lambda$, we have $\langle z_\beta \mid \beta < \theta \rangle \in L_\lambda[x]$. Since for each $\beta < \theta, z_\beta$ codes $(A \cap \eta_0^\beta, C \cap \eta_0^\beta)$, $(A \cap \gamma, C \cap \gamma) \in L_\lambda[x]$.

Subcase 1: $\alpha_\gamma \leq \lambda$. Since $\gamma, \eta^* \in D$ and $\lambda < \alpha_{\eta^*}$, we have $\gamma < \eta^*$. Note that for any $\eta < \eta^*$ with $\eta \in D$, $f_{\eta+i} : \omega \to \eta + i$ is the $L_{\alpha_\eta}[A \cap \eta, C \cap \eta]$-least surjection. So $\langle \delta_{\eta+i} \mid i \in \omega \rangle$ is Σ_1-definable in $L_{\alpha_\eta}[A \cap \eta, C \cap \eta]$ from $(A \cap \eta, C \cap \eta)$ for any $\eta \in D$. Since $\gamma \in D$, $\langle \delta_{\gamma+i} \mid i \in \omega \rangle$ is Σ_1-definable in $L_{\alpha_\gamma}[A \cap \gamma, C \cap \gamma]$ from $(A \cap \gamma, C \cap \gamma)$. So $\langle \delta_{\gamma+i} \mid i \in \omega \rangle$ is Σ_1-definable in $L_\lambda[x]$ from $(A \cap \gamma, C \cap \gamma)$. Since $(A \cap \gamma, C \cap \gamma) \in L_\lambda[x]$ and λ is x-admissible, $\langle \delta_{\gamma+i} \mid i \in \omega \rangle \in L_\lambda[x]$. Since $\omega \cdot \gamma = \gamma$, $z_\gamma = \{i \in \omega \mid \omega \cdot \gamma + i \in B\} = \{i \in \omega \mid \gamma + i \in B\} = \{i \in \omega \mid |x \cap \delta_{\gamma+i}| < \omega\}$, we have $z_\gamma \in L_\lambda[x]$. So $\gamma < \theta$. So $\gamma \geq \eta_0^\gamma$. Contradiction.

Subcase 2: $\lambda < \alpha_\gamma$. Since $A \cap \gamma \in L_\lambda[x]$, λ is $A \cap \gamma$-admissible. Since for any $\beta < \theta$, $z_\beta \in L_\lambda[x]$ and z_β codes η_0^β, we have $\gamma \leq \lambda$. Since $\gamma \in D$ and $\gamma \leq \lambda < \alpha_\gamma$, λ is an L-cardinal. $\square$

We have shown in $L[G][H][A][C]$ that $L_{\alpha_{\eta^*}}[x] \models Z_3$ + **Harrington's** $\star$. Now we show that $L_{\alpha_{\eta^*}}[x] \models 0^\sharp$ does not exist.

Lemma 2.3.58 $L_{\alpha_{\eta^*}}[x] \models 0^\sharp$ *does not exist.*

Proof Note that $\alpha_{\eta^*} < \delta^*$.[20] Suppose $L_{\alpha_{\eta^*}}[x] \models 0^\sharp$ exists. Let $\eta = \omega_1^{L_{\alpha_{\eta^*}}[x]}$. Since $L_{\alpha_{\eta^*}}[x] \cap \omega^\omega = L_\eta[x] \cap \omega^\omega$, we have $L_\eta[x] \models Z_2 + 0^\sharp$ exists. Note

[20] δ^* is defined in Section 2.3.3.

that $\eta < \alpha_{\eta^*} < \delta^*$. This contradicts that δ^* is the least α such that $\exists x \in \omega^\omega(L_\alpha[x] \models$ "$Z_2 + 0^\sharp$ exists"). $\square$

So we get a model $L_{\alpha_{\eta^*}}[x]$ such that $L_{\alpha_{\eta^*}}[x] \models$ "Z_3 + **Harrington's** $\star$ + $0^\sharp$ does not exist". Now we arrive at the following main theorem.

Theorem 2.3.59 *Z_3+ **Harrington's** $\star$ does not imply $0^\sharp$ exists.*

As a corollary of Theorem 2.3.59 and Theorem 2.0.3, we have:

Corollary 2.3.60 *Z_4 is the minimal system to show that **Harrington's** $\star$ implies $0^\sharp$ exists.*

As a summary, our proof shows that:

(i) $Con(Z_3+$ **Harrington's** $\star)$ implies $Con(Z_3+$ **Harrington's** $\star$ + $0^\sharp$ does not exist).

(ii) Suppose $L_\alpha[x] \models Z_2 + 0^\sharp$ exists for some $x \in \omega^\omega$ and some ordinal $\alpha < \omega_1$. Then $\exists x \in \omega^\omega\, \exists \beta < \alpha(L_\beta[x] \models$ "Z_3+ **Harrington's** $\star$ + $0^\sharp$ does not exist").

(iii) "$Z_2 + 0^\sharp$ exists" $\vdash Con(Z_3+$ **Harrington's** $\star$ + $0^\sharp$ does not exist).

(iv) If there exists a club on ω_2 of L-cardinals with weakly reflecting property, then we can construct a model of "Z_3+ **Harrington's** $\star$ + $0^\sharp$ does not exist".

Assuming $0^\sharp$ exists, we can force over L to get $L[G][H]$ such that in $L[G][H]$, any L-cardinal has weakly reflecting property and any L-cardinal $\alpha \leq \omega_2$ has strong reflecting property.

Question 2.3.61 *What is the large cardinal strength of the statement "any L-cardinal has weakly reflecting property".*

We conjecture that the large cardinal strength of the statement "any L-cardinal has weakly reflecting property" is strictly weaker than the existence of Erdös cardinal $\kappa(\omega)$ where $\kappa(\omega)$ is the least cardinal κ such that $\kappa \rightarrow (\omega)_2^{<\omega}$. So we can show $Con(Z_3+$ **Harrington's** $\star$ $+$ $0^\sharp$ does not exist) by assuming large cardinals compatible with L.

Chapter 3

Proof of Harrington's theorem

3.1 Boldface Harrington's theorem in Z_2

In this section, we prove boldface Martin-Harrington theorem in Z_2. Martin first proved in ZF that "$0^\sharp$ exists implies $Det(\Sigma^1_1)$" and "for any real $x, x^\sharp$ exists" implies $Det(\underset{\sim}{\Sigma}^1_1)$. We observe that Martin's theorems can be proved in Z_2.

Theorem 3.1.1 *(Martin, [9], [11]) Assume $Z_2 + 0^\sharp$ exists. Then for any Σ^1_1 game either Player I has a winning strategy or Player II has a winning strategy recursive in $0^\sharp$ (and hence a Δ^1_3 winning strategy).*

As a corollary, by relativization of Martin's theorem, we have $Z_2 + \forall x \in 2^\omega(x^\sharp$ exists) implies $Det(\underset{\sim}{\Sigma}^1_1)$.

Note that Z_2+ **Harrington's** $\star$ implies $\{\alpha < \omega_1 \mid \alpha$ is an L-cardinal$\}$ is a club on ω_1. If $M \models Z_2$, then either $L^M \models ZFC$ or $M \models$ "there exists a largest L-cardinal". Since Z_2+ **Harrington's** $\star$ implies that there is no largest L-cardinal, if $M \models Z_2+$ **Harrington's** $\star$, then $L^M \models ZFC$.

For any real y, **Harrington's** $\star(y)$ denotes the statement:

$$\exists x \in 2^\omega \forall \alpha < \omega_1(\alpha \text{ is } x\text{-admissible} \rightarrow \alpha \text{ is an } L[y]\text{-cardinal}).$$

Fix real y. Note that Z_2+ **Harrington's** $\star(y)$ implies that there is no largest $L[y]$-cardinal. So if $\varphi \in ZFC$, then Z_2+ **Harrington's** $\star(y) \vdash$ "$L[y] \models \varphi$".

Theorem 3.1.2 $Z_2 + Det(\underset{\sim}{\Sigma}^1_1)$ *implies for any real* $x, x^\sharp$ *exists.*[1]

Proof By relativizing Harrington's proof "$Z_2 + Det(\Sigma^1_1)$ implies **Harrington's** $\star$" to reals, we have $Z_2 + Det(\Sigma^1_1(x))$ implies **Harrington's** $\star(x)$ for any real x. So $Z_2+Det(\underset{\sim}{\Sigma}^1_1)$ implies for any real x, **Harrington's** $\star(x)$ holds.

Lemma 3.1.3 *Assume* $Z_2 + \forall y \in \omega^\omega$*(**Harrington's*** $\star(y)$*). Then for any real* $y, y^\sharp$ *exists.*

Proof Fix a real y. Let z be a witness real for **Harrington's** $\star(y)$ with $y \leq_T z$. Let s be a witness real for **Harrington's** $\star(z)$ with $z \leq_T s$. Since $Z_2 \vdash \forall \alpha < \omega_1 \forall x \in \omega^\omega$(the α-th countable x-admissible ordinal exists), there

[1] The idea of the proof is to show that "$Z_2 + \forall y \in \omega^\omega$(**Harrington's** $\star(y)$)" implies "for any real $y, y^\sharp$ exists". Fix a real y. To show that $y^\sharp$ exists, first let z be a witness real for **Harrington's** $\star(y)$ with $y \leq_T z$ and then let s be a witness real for **Harrington's** $\star(z)$ with $z \leq_T s$. Since $\omega_3^{L[z]}$ exists, $L_{\omega_3^{L[z]}}[z] \models Z_4+$ **Harrington's** $\star(y)$. Since Z_4+**Harrington's** $\star(y)$ implies $y^\sharp$ exists, $L_{\omega_3^{L[z]}}[z] \models y^\sharp$ exists. By absoluteness, $y^\sharp$ exists.

exists an increasing sequence $\langle \lambda_i \mid i \in \omega \rangle$ of countable s-admissible ordinals. So for any $i \in \omega, \lambda_i$ is an $L[z]$-cardinal. So $\omega_3^{L[z]}$ exists. Let $\lambda = \omega_3^{L[z]}$. From the definition of λ and Z_4, we have $L_\lambda[z] \models Z_4$. Note that **Harrington's** $\star(y)$ is $\Sigma^1_3(y)$. Since z witnesses **Harrington's** $\star(y)$ and $y \leq_T z$, we have $L_\lambda[z] \models$ **Harrington's** $\star(y)$. From [18], by relativizing Theorem 2.0.3 to real y, we have Z_4+**Harrington's** $\star(y)$ implies $y^\sharp$ exists. So $L_\lambda[z] \models y^\sharp$ exists.

Fact 3.1.4 *([11], [9])*

(a) $Z_2 \vdash \forall x \in \omega^\omega \, \forall y \in \omega^\omega$ *(if* $\omega_1^{L[x]}$ *exists and* $L_\lambda[x] \models$ *"*$y^\sharp$ *exists" for some* $\lambda \geq \omega_1^{L[x]}$*, then* $L[x] \models$ *"*$y^\sharp$ *exists").*

(b) $Z_2 \vdash \forall x \in \omega^\omega \, \forall y \in \omega^\omega$ *(if* $L[x] \models y^\sharp$ *exists, then* $y^\sharp$ *exists).*

Note that "$y^\sharp$ exists" is $\Sigma^1_3(y)$. Since $\omega_1^{L[z]} < \lambda, L[z] \models y^\sharp$ exists. By absoluteness, $y^\sharp$ exists. $\square$

$\square$

Corollary 3.1.5 (Z_2) *$Det(\undertilde{\Sigma}^1_1)$ if and only if for any real $x, x^\sharp$ exists.*

3.2 W.Hugh Woodin's proof of Harrington's theorem

In this section, we present W.Hugh Woodin's proof of "$Det(\Sigma^1_1)$ implies **Harrington's** $\star$" in Z_2.

Definition 3.2.1 *(W.Hugh Woodin) Consider the following game G:*

$$\begin{array}{c|cc} I & x & \\ \hline II & & y \end{array}$$

in which $x \in \omega^\omega$ codes a linear ordering on ω and $y \in \omega^\omega$ codes an ω-model M of $ZFC^- + V = L$. Player II wins G iff if x codes a well ordering on ω, say of order type α, then there exists π such that π embeds α into an initial segment of Ord^M.

Note that the winning condition for Player II is Σ^1_1 and the winning condition for Player I is Π^1_1. So G is a Π^1_1 game.

Lemma 3.2.2 *Player I does not have a winning strategy in G.*

Proof Let σ be the winning strategy for player I. For any $y \in 2^\omega$, $(\omega, E_{(\sigma * y)_I})$ is a well ordering on ω. Note that $\{(\sigma * y)_I \mid y \in 2^\omega\}$ is $\Sigma^1_1(\sigma)$. So there exists a countable ordinal α such that $o.t.(\omega, E_{(\sigma*y)_I}) < \alpha$ for all $y \in 2^\omega$. Take an ω-model (ω, E) of $ZFC^- + V = L$ such that $osp((\omega, E)) > \alpha$. Choose $y \in 2^\omega$ such that y codes (ω, E). Let player II play y against σ. This defeats σ. $\square$

Lemma 3.2.3 *Suppose $\mathcal{M} = (M, E)$ is a countable ω-model of ZFC^- and $osp(\mathcal{M}) = \alpha$. Suppose*

(a) $a \in Ord^{\mathcal{M}}$, $a > osp(\mathcal{M})$ and

(b) $\forall b \in Ord^{\mathcal{M}}(b\, E\, a \to b + a = a)$.

Then

$$(\{b \in M \mid b\,E\,a\}, E) \cong (\alpha + \alpha \cdot \mathbb{Q}, \in).$$

Proof For $c, d \in Ord^{\mathcal{M}}$ with $c\,E\,d$, define $c \sim d$ if for some $\beta < \alpha, c + \beta = d$. For $c \in Ord^{\mathcal{M}}$, let $[c] = \{a \in Ord^{\mathcal{M}} \mid a \sim c\}$.

Claim If $\mathcal{M}$ is an ω-model of ZFC^-, then for any $d \in Ord^{\mathcal{M}}$, $\{d - c \mid c\,E\,d\}$ is finite.

Proof If not, then there exists a descending E-sequence of ordinals from $\mathcal{M}$. □

Claim For any $c \in Ord^{\mathcal{M}}$, $[c]$ has an E-least element.

Proof It suffices to show that if $d \in Ord^{\mathcal{M}}$ and $d \notin osp(\mathcal{M})$, then $[d]$ has an E-least element.

$$[d] = \{e \mid e\,E\,d \wedge d - e < \alpha\} = \{e \mid e\,E\,d \wedge d - e \in S\}$$

where $S = \{d - e \mid e\,E\,d\} \cap \alpha$.

Since $\{d - e \mid e\,E\,d\}$ is finite, S is finite. Since S is finite and $osp(\mathcal{M}) = \alpha$, $[d]$ is definable in $\mathcal{M}$ with parameters and for any $e \in [d], e > \alpha$. If $[d]$ does not have an E-least element, then there exists an E-descending sequence $\langle e_n : n \in \omega \rangle$ from $[d]$ converging to α. Since any element of $[d]$ is definable in $\mathcal{M}$ with parameters, α is definable in $\mathcal{M}$ with parameters. Contradiction. So $[d]$ has an E-least element. □

Proposition 3.2.4 *If $a, b \in Ord^{\mathcal{M}}, a\, E\, b$ and $a \not\approx b$, then*

$$\exists c \in Ord^{\mathcal{M}}(a\, E\, c \wedge c\, E\, b \wedge a \not\approx c \wedge c \not\approx b).$$

Proof Let $A = \{d \in Ord^{\mathcal{M}} \mid \mathcal{M} \models a < a + d < a + d + d < b\}$. Since $\alpha = osp(\mathcal{M}), a\, E\, b$ and $a \not\approx b$, we have $A \supseteq \alpha$. Since α is not definable in $\mathcal{M}$, $A \nsubseteq \alpha$. So there is $d \in A$ such that $d > \alpha$. Let $c = a + d$. Since $d \in A$, $a\, E\, c$ and $c\, E\, b$. It is easy to check that $a \not\approx c$ and $c \not\approx b$. (If $c + \beta = b$ for some $\beta < \alpha$, then $\mathcal{M} \models$ "$b = c + \beta = a + d + \beta < a + d + d < b$". Contradiction.) $\square$

Now for cEa such that $c \notin osp(\mathcal{M}), [c] = \{d + \beta \mid \beta < \alpha\}$ where $d \in [c]$ is the E-least element. Let

$$X = \{[c] \mid c\, E\, a \wedge c > \alpha\}.$$

For $[d], [e] \in X$, define $[d] < [e] \leftrightarrow (d\, E\, e \wedge d \not\approx e)$. We show that X is a countable dense order without endpoints. Since $\mathcal{M}$ is countable, X is countable. By Proposition 3.2.4, $(X, <)$ is a dense order. Suppose $c\, E\, a$ and $c > \alpha$. Take any β such that $\beta\, E\, \alpha$. By Proposition 3.2.4, $\exists d \in Ord^{\mathcal{M}}(\beta\, E\, d \wedge d\, E\, c \wedge \beta \not\approx d \wedge d \not\approx c)$. Let $d \in Ord^{\mathcal{M}}$ be a witness such that $\beta\, E\, d, d\, E\, c, \beta \not\approx d$ and $d \not\approx c$. Since $d\, E\, c$ and $c\, E\, a$, we have $d\, E\, a$. Since $d\, E\, c$ and $d \not\approx c$, we have $[d] < [c]$. Since $\beta\, E\, d, \beta\, E\, \alpha$ and $\beta \not\approx d$, we have $d > \alpha$. Since $c\, E\, a$, by Proposition 3.2.4, $\exists d \in Ord^{\mathcal{M}}([c] < [d] \wedge d\, E\, a \wedge d > \alpha)$. So $(X, <)$ has no endpoints. Hence $(X, <) \cong (\mathbb{Q}, \in)$ and $(\{b \in M \mid b\, E\, a\}, E) \cong (\alpha + \alpha \cdot \mathbb{Q}, \in)$.

$\square$

Corollary 3.2.5 *Let $\mathcal{M} = (M, E)$ be an ill-founded countable ω-model of ZFC^- and $\alpha = osp(\mathcal{M})$. Then $(Ord^{\mathcal{M}}, E) \cong (\alpha + \alpha \cdot \mathbb{Q}, \in)$.*

Theorem 3.2.6 *Assume $Z_2 + Det(G)$. Suppose τ is the winning strategy for player II in game G. Then if λ is a countable τ-admissible ordinal, then λ is an L-cardinal. So **Harrington's** $\star$ holds.*

Proof Since "τ is a winning strategy for player II in G" is Π^1_2, for any ω-model M with $\tau \in (\omega^\omega)^M$, τ is a winning strategy for player II in G if and only if $M \models$ "τ is a winning strategy for player II in G".

Suppose λ is a countable τ-admissible ordinal. We show that λ is an L-cardinal. Suppose λ is not an L-cardinal and we try to get a contradiction. Let $\delta = |\lambda|^L$. Since λ is not an L-cardinal, $\delta < \lambda$.

Fact 3.2.7 *(J. Barwise, [1], [18])*

(i) Suppose $x \in \omega^\omega, \lambda < \omega_1$ is x-admissible, $\delta < \lambda, \delta$ is an L-cardinal and $(\delta^+)^L \geq \lambda$. Suppose $\varphi(x)$ is a formula in $\mathfrak{L}_{st}$ and there exists a transitive set N such that $x \in N$ and $N \models \varphi[x]$. Then there exists a countable ω-model $\mathcal{M}$ such that $\mathcal{M}$ is not well founded, $osp(\mathcal{M}) = \lambda, x \in \mathcal{M}, \mathcal{M} \models \varphi[x], \mathcal{M} \models$ "δ is an L-cardinal" and for any $\alpha < \lambda, \mathcal{M} \models (\delta^+)^L > \alpha$.

(ii) *Suppose* $\mathcal{M}$ *is a countable* ω*-model,* $\mathcal{M}$ *is not well founded,* $y \in wfp(\mathcal{M})$*,* $\varphi(x)$ *is a formula in* $\mathfrak{L}_{st}$ *and there exists a transitive set* E *such that* $y \in E$ *and* $E \models \varphi[y]$*. Then there exists a countable* ω*-model* $\mathcal{N}$ *such that* $\mathcal{N}$ *is not well founded,* $y \in wfp(\mathcal{N}), \mathcal{N} \models \varphi[y]$ *and* $osp(\mathcal{N}) = osp(\mathcal{M})$*.*

Lemma 3.2.8 *There exist countable* ω*-models* $\mathcal{M}_1 = (M_1, E_1)$ *and* $\mathcal{M}_2 = (M_2, E_2)$ *such that*

(1) $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *are not well founded;*

(2) $osp(\mathcal{M}_1) = osp(\mathcal{M}_2) = \lambda, \mathcal{M}_1 \models ZFC^-$ *and* $\mathcal{M}_2 \models ZFC^-$*;*

(3) $\tau \in \mathcal{M}_1, \tau \in \mathcal{M}_2, \mathcal{M}_1 \models$ *"*τ *is a winning strategy for player II in* G*" and* $\mathcal{M}_2 \models$ *"*τ *is a winning strategy for player II in* G*";*

(4) $\mathcal{M}_1 \models$ *"*δ *is an L-cardinal" and* $\mathcal{M}_2 \models$ *"*δ *is an L-cardinal";*

(5) Let $a_1 \in Ord^{\mathcal{M}_1}$ *be such that* $\mathcal{M}_1 \models a_1 = (\delta^+)^L$*. Then for any* $\alpha < \lambda, \mathcal{M}_1 \models \alpha < a_1$*;*

(6) Let $a_2 \in Ord^{\mathcal{M}_2}$ *be such that* $\mathcal{M}_2 \models a_2 = (\delta^+)^L$*. Then for any* $\alpha < \lambda, \mathcal{M}_2 \models \alpha < a_2$*;*

(7) $(L_{a_1})^{\mathcal{M}_1} \ncong (L_b)^{\mathcal{M}_2}$ *for any* $b \leq a_2$ *with* $b \in \mathcal{M}_2$*;*

(8) $(L_{a_2})^{\mathcal{M}_2} \ncong (L_b)^{\mathcal{M}_1}$ *for any* $b \leq a_1$ *with* $b \in \mathcal{M}_1$*.*

Proof By Fact 3.2.7(1), there exists a countable ω-model $\mathcal{M}_1$ such that $\mathcal{M}_1$ is not well founded, $osp(\mathcal{M}_1) = \lambda, \tau \in \mathcal{M}_1, \mathcal{M}_1 \models$ "δ is an L-cardinal" and $\mathcal{M}_1 \models$ "$ZFC^- + \tau$ is a winning strategy for player II in game G". Let $a_1 \in Ord^{\mathcal{M}_1}$ be such that $\mathcal{M}_1 \models a_1 = (\delta^+)^L$. Then for any $\alpha < \lambda, \mathcal{M}_1 \models \alpha < a_1$. Without loss of generality, we can assume that a_1 is countable in $\mathcal{M}_1$ (if not, we can replace $\mathcal{M}_1$ by $\mathcal{M}_1[G]$ where $G \subseteq Col(\omega, a_1)$ is $Col(\omega, a_1)$-generic over $\mathcal{M}_1$). Choose $y \subseteq \omega$ in $\mathcal{M}_1$ which codes $L_{a_1}^{\mathcal{M}_1}$. Such y exists since $\mathcal{M}_1 \models$ "a_1 is countable".

Apply Fact 3.2.7(2) to $\mathcal{M}_1$ with parameters (δ, τ, y) since $(\delta, \tau, y) \in wfp(\mathcal{M}_1)$. Then there exists a countable ω-model $\mathcal{M}_2$ such that

(a) $\mathcal{M}_2$ is not well founded, $\{\tau, \delta, y\} \subseteq \mathcal{M}_2$;

(b) $osp(\mathcal{M}_2) = \lambda, \mathcal{M}_2 \models$ "$ZFC^- + \tau$ is a winning strategy for player II in game G";

(c) $\mathcal{M}_2 \models$ "δ is an L-cardinal";

(d) Let $a_2 \in Ord^{\mathcal{M}_2}$ be such that $\mathcal{M}_2 \models a_2 = (\delta^+)^L$. Then for any $\alpha < \lambda, \mathcal{M}_2 \models \alpha < a_2$;

(e) $\mathcal{M}_2 \models$ "y codes a non-well founded model (M, E) and L_{a_2} is not isomorphic to an initial segment of (M, E)". i.e. $\mathcal{M}_2 \models$ "y codes a non-well founded model (M, E) and $osp((M, E)) \leq a_2$".

Note that we can take such $\mathcal{M}_2$ with condition (e) since in V, y codes a non-well founded ω-model with ordinal standard part $\leq ((\delta^+)^L)$.

Claim $(L_{a_1})^{\mathcal{M}_1} \ncong (L_b)^{\mathcal{M}_2}$ for any $b \leq a_2$ with $b \in \mathcal{M}_2$.

Proof Suppose not. Assume $b \leq a_2, b \in \mathcal{M}_2$ and $\pi : (L_{a_1})^{\mathcal{M}_1} \cong (L_b)^{\mathcal{M}_2}$. Note that $\pi \upharpoonright (\delta+1) = id$. So $\pi \upharpoonright (\mathcal{P}(\delta) \cap (L_{a_1})^{\mathcal{M}_1}) = id$. So π is unique. Then $\mathcal{M}_2 \models \exists b \leq a_2((M, E) \cong L_b)$. So $\mathcal{M}_2 \models y$ codes a well founded model. Contradiction. $\square$

Claim $(L_{a_2})^{\mathcal{M}_2} \ncong (L_b)^{\mathcal{M}_1}$ for any $b \leq a_1$ with $b \in \mathcal{M}_1$.

Proof Suppose not. Assume that $(L_{a_2})^{\mathcal{M}_2} \cong (L_b)^{\mathcal{M}_1}$ for some $b \leq a_1$ with $b \in \mathcal{M}_1$. Note that $\pi \upharpoonright (\delta+1) = id$. So $\pi \upharpoonright (\mathcal{P}(\delta) \cap (L_{a_2})^{\mathcal{M}_2}) = id$. So $\mathcal{M}_2 \models$ "L_{a_2} is isomorphic to an initial segment of (M, E)". Hence $\mathcal{M}_2 \models$ "$a_2 \leq osp((M, E))$". Since $\mathcal{M}_2 \models y$ codes a non-well founded model (M, E) and $osp((M, E)) \leq a_2$, we have $\mathcal{M}_2 \models a_2 = osp((M, E))$. Then $\mathcal{M}_2 \models \exists b \in Ord^{\mathcal{M}_2}(L_{a_2} \cong (L_b)^{(M,E)})$. So $\mathcal{M}_2 \models a_2 + 1 \leq osp((M, E))$. But $\mathcal{M}_2 \models osp((M, E)) \leq a_2$. Contradiction. $\square$

So we have shown that $\mathcal{M}_1$ and $\mathcal{M}_2$ satisfy the conditions (7) and (8). This finishes the proof. $\square$

Fix ω-models $\mathcal{M}_1$ and $\mathcal{M}_2$ as in Lemma 3.2.8. Define

$$\gamma_1 = (\delta^+)^{L^{\mathcal{M}_1}} \quad \text{and} \quad \gamma_2 = (\delta^+)^{L^{\mathcal{M}_2}}.$$

Since $\delta = |\lambda|^L$, $\gamma_1 \geq \lambda$ and $\gamma_2 \geq \lambda$.

By Lemma 3.2.3, we have

$$(\{b \in M_1 \mid b\, E_1\, \gamma_1\}, E_1)^{\mathcal{M}_1} \cong (\{b \in M_2 \mid b\, E_2\, \gamma_2\}, E_2)^{\mathcal{M}_2} \cong (\lambda + \lambda \times \mathbb{Q}, \in).$$

Fix $\pi : (\{b \in M_1 \mid b\, E_1\, \gamma_1\}, E_1)^{\mathcal{M}_1} \cong (\{b \in M_2 \mid b\, E_2\, \gamma_2\}, E_2)^{\mathcal{M}_2}$. Let $Col(\omega, \eta) = \{s \mid s : n \to \eta$ for some $n < \omega$ and s is injective$\}$ with order by extension. So we have

$$Col(\omega, \{b \mid b\, E_1\, \gamma_1\})^{\mathcal{M}_1} \cong Col(\omega, \{b \mid b\, E_2\, \gamma_2\})^{\mathcal{M}_2} \cong Col(\omega, \lambda + \lambda \times \mathbb{Q}).$$

Let g_1 be $Col(\omega, \{b \mid b\, E_1\, \gamma_1\})^{\mathcal{M}_1}$-generic over $\mathcal{M}_1$ and g_2 be $Col(\omega, \{b \mid b\, E_2\, \gamma_2\})^{\mathcal{M}_2}$-generic over $\mathcal{M}_2$ such that $\pi(g_1) = g_2$. Define

$$I_{g_1} = \{(i,j) \mid g_1(i)\, E_1\, g_1(j)\} \text{ and } I_{g_2} = \{(i,j) \mid g_2(i)\, E_2\, g_2(j)\}.$$

Since $\pi(g_1) = g_2, I_{g_1} = I_{g_2}$. From the definition of I_{g_1} and I_{g_2}, $I_{g_1} \in \mathcal{M}_1[g_1]$ and $I_{g_2} \in \mathcal{M}_2[g_2]$. Let $x \subseteq \omega$ codes $I_{g_1} = I_{g_2}$. Then $x \in \mathcal{M}_1[g_1] \cap \mathcal{M}_2[g_2]$. By absoluteness, $\mathcal{M}_1[g_1] \models$ "τ is a winning strategy for player II in G" and $\mathcal{M}_2[g_2] \models$ "τ is a winning strategy for player II in G". Note that

(1) $\mathcal{M}_1[g_1] \models$ "x codes a well order isomorphic to γ_1" and

(2) $\mathcal{M}_2[g_2] \models$ "x codes a well order isomorphic to γ_2".

Let player I play real x and let $y = (x * \tau)_{II}$. So

(a) $\mathcal{M}_1[g_1] \models$ "y codes an ω-model M of $ZFC^-+V = L$ such that $osp(M) \geq \gamma_1$" and

(b) $\mathcal{M}_2[g_2] \models$ "y codes an ω-model M of $ZFC^-+V = L$ such that $osp(M) \geq \gamma_2$".

Let $\mathcal{N} = (N, E)$ be the model coded by y. So

(i) there exists $b_1 \in Ord^{\mathcal{N}}$ such that $\mathcal{M}_1[g_1] \models b_1 \cong \gamma_1$ and

(ii) there exists $b_2 \in Ord^{\mathcal{N}}$ such that $\mathcal{M}_2[g_2] \models b_2 \cong \gamma_2$.

We know that $\mathcal{M}_1, \mathcal{M}_2$ have the following property:

$$(L_{a_1})^{\mathcal{M}_1} \ncong (L_b)^{\mathcal{M}_2} \text{ for any } b \leq a_2 \text{ with } b \in \mathcal{M}_2 \text{ and } (L_{a_2})^{\mathcal{M}_2} \ncong (L_b)^{\mathcal{M}_1} \text{ for any } b \leq a_1 \text{ with } b \in \mathcal{M}_1.$$

Case 1: $b_1 \, E \, b_2$. Since $(L_{a_1})^{\mathcal{M}_1} \cong (L_{b_1})^{\mathcal{N}}$ and $(L_{a_2})^{\mathcal{M}_2} \cong (L_{b_2})^{\mathcal{N}}$, we have $(L_{a_1})^{\mathcal{M}_1} \cong (L_b)^{\mathcal{M}_2}$ for some $b \leq a_2$ with $b \in \mathcal{M}_2$. Contradiction.

Case 2: $b_2 \, E \, b_1$. Similarly, we have $(L_{a_2})^{\mathcal{M}_2} \cong (L_b)^{\mathcal{M}_1}$ for some $b \leq a_1$ with $b \in \mathcal{M}_1$. Contradiction.

Case 3: $b_1 = b_2$. Then $(L_{a_1})^{\mathcal{M}_1} \cong (L_{a_2})^{\mathcal{M}_2}$. Contradiction. □

Corollary 3.2.9 *$Z_4 + Det(\Sigma^1_1)$ implies $0^\sharp$ exists.*

Proof Z_4+ **Harrington's** $\star$ implies $0^\sharp$ exists and $Z_2 + Det(\Sigma^1_1)$ implies **Harrington's** $\star$. □

Chapter 4

Conclusions

In this thesis, we have proved that Z_2+**Harrington's** $\star$ does not imply $0^\sharp$ exists and Z_3+**Harrington's** $\star$ does not imply $0^\sharp$ exists. As a corollary of "Z_4+**Harrington's** $\star$ implies $0^\sharp$ exists", Z_4 is the minimal system to prove "**Harrington's** $\star$ implies $0^\sharp$ exists".

Harrington first raised the question whether "$Det(\underset{\sim}{\Sigma}^1_1)$ implies for all real $x, x^\sharp$ exists" can be proved in analysis. From Chapter 3, the answer is yes. In fact, we observe that all known equivalences in ZFC between determinacy and "$\forall x \in 2^\omega(x^\sharp \text{ exists})$" are provable in Z_2.

The status of lightface Harrington's theorem is more subtle than boldface Harrington's theorem. We only know that $Z_4 + Det(\Sigma^1_1)$ implies $0^\sharp$ exists. We do not know that whether Z_4 is the minimal system to prove "$Det(\Sigma^1_1)$ implies $0^\sharp$ exists". So the next question is:

Question 4.0.10 *Whether $Z_2 + Det(\Sigma^1_1)$ implies $0^\sharp$ exists?*

If the answer for Question 4.0.10 is yes, then the lightface Martin-Harrington theorem "$Det(\Sigma^1_1) \leftrightarrow 0^\sharp$ exists" can be proved in analysis (Z_2).

Since we have shown that Z_3+ **Harrington's** $\star$ does not imply $0^\sharp$ exists, if "$Z_3 + Det(\Sigma^1_1)$ implies $0^\sharp$ exists" or "$Z_2 + Det(\Sigma^1_1)$ implies $0^\sharp$ exists" holds, then there must be a new and different proof of "$Det(\Sigma^1_1)$ implies $0^\sharp$ exists" without the use of Silver's theorem which derives the existence of $0^\sharp$ from **Harrington's** $\star$.

Let G be the Π^1_1 game defined by W.Hugh Woodin as in Definition 3.2.1. We know that Player I has no winning strategy in game G. We make the following conjecture.

Conjecture 4.0.11 *(W.Hugh Woodin)* (Z_2) *If Player II has a winning strategy in game G, then $0^\sharp$ exists.*

If this conjecture is true, then $Z_2+Det(\Pi^1_1)$ implies $0^\sharp$ exists which answers Question 4.0.10.

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